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A nonlinear Fokker–Planck equation approach for interacting systems: Anomalous diffusion and Tsallis statistics

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ABSTRACT

We investigate the solutions for a set of coupled nonlinear Fokker–Planck equations coupled by the diffusion coefficient in presence of external forces. The coupling by the diffusion coefficient implies that the diffusion of each species is influenced by the other and vice versa due to this term, which represents an interaction among them. The solutions for the stationary case are given in terms of the Tsallis distributions, when arbitrary external forces are considered. We also use the Tsallis distributions to obtain a time dependent solution for a linear external force. The results obtained from this analysis show a rich class of behavior related to anomalous diffusion, which can be characterized by compact or long-tailed distributions.

1. Introduction

The botanist Robert Brown gave an important step to understanding several phenomena in science when discovering, under his microscope, irregular motion of small particles originating from the pollen floating in water [1]. This motion was also detected in different scenarios drawing the attention of several researchers, who have developed a theory to explain it by combining such motion with the molecular motion. In particular, the first theoretical explanation for this phenomenon was proposed by A. Einstein in 1905 [2], which was followed by the contributions of P. Langevin, M. Smoluchowski, G. U. Uhlenbeck, and many others. One of the main points of this motion is the linear time dependence for the mean square displacement, i.e., $\langle (x-\langle x\rangle)^2\rangle \sim t$, which is directly related to the Markovian characteristic of this motion, namely diffusion. However, a large number of experimental scenarios such as electrical response [3,4], transport in porous media [5], diffusion on fractals [6], transport in amorphous semiconductors [7], and heat distribution in materials [8], superconductors [9], and transport of cosmic rays [10] have shown that the diffusion may be anomalous and characterized by a nonlinear time dependence on the mean square displacement, e.g., $\langle (x-\langle x\rangle)^2\rangle \sim t^{\alpha}$, ($\alpha < 1$ and > 1 corresponds to sub and superdiffusion, respectively) where non-Markovian characteristics are present. These scenarios have been investigated by different approaches such as fractional diffusion equations [11–14], random walks [15–17], master equations [18], generalized Langevin equations, and nonlinear diffusion equations [19–21]. It is worth mentioning that the last approach can be connected with a thermostatistics context [22,23] characterized by distributions with a compact or long-tailed behavior, where the H-theorem has been used to establish the entropy behind the dynamic of these systems [24–26] and the entropic additivity [27].

In these processes, interactions among the particles may also occur in addition to the diffusive phenomena modifying the relaxation, i.e., the spreading of the system during the time evolution. A typical scenario is observed, for example, in chemical reactions, in which the interaction between particles can lead to the conversion of one chemical species into another. Another scenario consists in considering a coupling among the system of particles, which compose the system in such way that each of their species may act like a thermal bath to the other during the relaxation process and vice versa due to this interaction. A situation related to this scenario emerges from a system composed of two subsystems, for instance, 1 and 2, with the particles governed by the nonlinear Langevin equations

$$\dot{x}_1 = F_1(x_1,t) + \sqrt{2\mathcal{D}_1(t)\rho_1^{\nu_1-1}(x_1,t)}\xi_1(t) \tag{1}$$

and

$$\dot{x}_2 = F_2(x_2,t) + \sqrt{2\mathcal{D}_2(t)\rho_2^{\nu_2-1}(x_2,t)}\xi_2(t)\,. \tag{2}$$

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In Eqs. (1) and (2), $F_1(x_2,t)$ and $F_2(x_2,t)$ are external forces, $\xi_{1(2)}(t)$ is a stochastic force (in this case, white noise with $\langle\xi_1(t')\xi_2(t)\rangle = 0$, i.e., uncorrelated) and the diffusion coefficients are given by

$$\mathcal{D}_1(t) = D_1 \int_{-\infty}^{\infty} dx_2 \rho_2^{\nu_2}(x_2,t) \tag{3}$$

and

$$\mathcal{D}_2(t) = D_2 \int_{-\infty}^{\infty} dx_1 \rho_1^{\nu_1}(x_1,t) \,, \tag{4}$$

where $\rho_1(x_1,t)$ and $\rho_2(x_2,t)$ represent the distributions related to each subsystem. Note that, by using the Tsallis entropy,

$$S_\nu^{(i)} = \frac{1}{\nu-1}\left(1 - \int_{-\infty}^{\infty} dx_i \rho_i^{\nu_i}(x_i,t)\right) \quad (\text{with } \; i=1,2) \,, \tag{5}$$

it is possible to rewrite Eqs. (3) and (4) as follows: $\mathcal{D}_1(t) = D_1(1-(\nu_2-1)S_{\nu_2}^{(2)}(t))$ and $\mathcal{D}_2(t) = D_2(1-(\nu_1-1)S_{\nu_1}^{(1)}(t))$. These results show that the coupling between the systems 1 and the 2 depends on the entropy of each other and by using the Einstein–Smoluchowski relation, a connection between the diffusion coefficient and temperature, i.e., $\mathcal{D} \propto T$ may be established. This feature implies that the time variation of the temperature of one these systems is directly related to the entropy production of the other since that $\dot{\mathcal{D}} \propto \dot{\mathcal{S}} \propto \dot{T}$. Thus, each subsystem may work like a thermal bath for the other by influencing the motion of the particles and, consequently, modifying the relation process due to the coupling, which promotes an interaction among the parts of the system. In this scenario, the anomalous diffusion may also be produced by a coupling among these different systems which are diffusing in the media. Equations (1) and (2) may be related to the macroscopic equations for the density of probability by using the formalism presented in Ref. [28]. In particular, after some calculations, we may obtain the nonlinear Fokker–Planck related to these Langevin equations. They are given by

$$\frac{\partial}{\partial t}\rho_1(x_1,t) = \mathcal{D}_1(t)\frac{\partial^2}{\partial x_1^2}\rho_1^{\nu_1}(x_1,t) - \frac{\partial}{\partial x_1}\left[F_1(x_1,t)\rho_1(x_1,t)\right], \tag{6}$$

$$\frac{\partial}{\partial t}\rho_2(x_2,t) = \mathcal{D}_2(t)\frac{\partial^2}{\partial x_2^2}\rho_2^{\nu_2}(x_2,t) - \frac{\partial}{\partial x_2}\left[F_2(x_2,t)\rho_2(x_2,t)\right]. \tag{7}$$

These equations are similar to the ones used in Ref. [27] to investigate the entropic additivity and, depending on the external force considered, they may present stationary solutions.

Our goal is to investigate the behavior of the solutions for these coupled nonlinear Fokker–Planck equations in order to analyze the effect of each one on the other by taking the relation process into account. This feature will be evidenced by the behavior of the mean square displacement which is directly related to the relaxation process of each system. In this scenario, we also investigate the entropy production for the system composed of the subsystems, with the dynamic governed by Eqs. (3) and (4), by considering two different entropic additivities. These analyses are performed in Sec. 2 and in Sec. 3 our conclusions are presented.

2. Nonlinear Fokker–Planck equations

Let us start our analysis with the solutions of Eqs. (6) and (7) by considering that the external forces can be written as $F_1(x_1,t) = -\partial_{x_1}V_1(x_1,t)$ and $F_2(x_2,t) = -\partial_{x_2}V_2(x_2,t)$, where $V_1(x_1,t)$ and $V_2(x_2,t)$ have, at least, one point of minimum each. In this case, for long times, the distributions reach a stationary behavior, i.e., they are time independent in this limit. Equations (6) and (7) subjected to these external forces and the boundary conditions $\rho_1(\pm\infty,t)=0$ and $\rho_1(\pm\infty,t)=0$ can be simplified to the following equations

$$\mathcal{D}_{1,\infty}\frac{\partial}{\partial x_1}\rho_{1,st}^{\nu_1}(x_1) - F_{1,\infty}(x_1)\rho_{1,st}(x_1) = 0, \tag{8}$$

$$\mathcal{D}_{2,\infty}\frac{\partial}{\partial x_2}\rho_{2,st}^{\nu_2}(x_2) - F_{2,\infty}(x_2)\rho_{2,st}(x_2) = 0, \tag{9}$$

where $lim_{t\to\infty}\mathcal{D}_{1(2)}(t) = \mathcal{D}_{1(2),\infty} = const$, $lim_{t\to\infty}\rho_{1(2)}(x_{1(2)},t) = \rho_{1(2),st}(x_{1(2)})$, and $lim_{t\to\infty}F_{1(2)}(x_{1(2)},t) = F_{1(2),\infty}(x_{1(2)})$. The solutions for previous time independent equations can be found by performing some integrations. In particular, they are

$$\rho_{1,st}(x_1) = \exp_{q_1}\left[-\beta_{1,st}V_{1,\infty}(x_1)\right] \Big/ \mathcal{Z}_{1,st} \,, \tag{10}$$

$$\rho_{2,st}(x_2) = \exp_{q_2}\left[-\beta_{2,st}V_{2,\infty}(x_2)\right] \Big/ \mathcal{Z}_{2,st} \,, \tag{11}$$

with $\beta_{1(2),st} = \mathcal{Z}_{1(2),st}^{\nu_{1(2)}-1}/[(2-q_{1(2)})\mathcal{D}_{1(2),\infty}]$ and $\nu_{1(2)} = 2-q_{1(2)}$, where the quantities $\mathcal{Z}_{1,st}$ and $\mathcal{Z}_{2,st}$ are related by Eqs. (3) and (4), and

$$\exp_{q_{1(2)}}\left[x_{1(2)}\right] \equiv \begin{cases} (1+(1-q_{1(2)})x_{1(2)})^{\frac{1}{1-q_{1(2)}}} & , \; x_{1(2)} > 1/(1-q_{1(2)}), \\ 0 & , \; x_{1(2)} < 1/(1-q_{1(2)}). \end{cases} \tag{12}$$

The presence of the q-exponential in Eqs. (10) and (11) enables us to obtain a short ($q_{1(2)} < 1$) or a long ($q_{1(2)} > 1$) tailed behavior for the solution depending on the choice of the parameter q. In fact, Equation (12) has a compact behavior for $q_{1(2)}$ less than one due to the "cut-off" manifested by the q-exponential to retain the probabilistic interpretation associated to $\rho_1(x,t)$. Consequently, $\rho_2(x,t)$ exhibits a similar behavior for q less than one. On the other hand, for $q_{1(2)}$ greater than one, Eqs. (15) and (16) have the asymptotic limit governed by a power–law behavior, which may be also related to a Lévy distribution, as shown in [29,30]. In this case, the solutions obtained for the previous set of equations may be asymptotically related to the solutions obtained in [31] for fractional diffusion equations, which asymptotically are governed by power laws as performed in Refs. [32,33,11].

Now, we consider the external forces given by $F_1(x_1,t) = -k_1(t)x_1$ and $F_2(x_2,t) = -k_2(t)x_2$, where $k(t)$ is an arbitrary time dependent function and the same boundary conditions. In this case, we can write Eqs. (6) and (7) as follows:

$$\frac{\partial}{\partial t}\rho_1(x_1,t) = \mathcal{D}_1(t)\frac{\partial^2}{\partial x_1^2}\rho_1^{\nu_1}(x_1,t) + k_1(t)\frac{\partial}{\partial x_1}\left(x_1\rho_1(x_1,t)\right), \tag{13}$$

$$\frac{\partial}{\partial t}\rho_2(x_2,t) = \mathcal{D}_2(t)\frac{\partial^2}{\partial x_2^2}\rho_2^{\nu_2}(x_2,t) + k_2(t)\frac{\partial}{\partial x_2}\left(x_2\rho_2(x_2,t)\right). \tag{14}$$

We consider, for this case, that the solutions are:

$$\rho_1(x_1,t) = \exp_{q_1}\left[-\beta_1(t)x_1^2\right] \Big/ \mathcal{Z}_1(t) \,, \tag{15}$$

$$\rho_2(x_2,t) = \exp_{q_2}\left[-\beta_2(t)x_2^2\right] \Big/ \mathcal{Z}_2(t) \,, \tag{16}$$

where the time dependent parameters $\beta_1(t)$, $\beta_1(t)$, $\mathcal{Z}_1(t)$, and $\mathcal{Z}_2(t)$ are obtained by substituting the proposed solutions (15) and (16) in (6) and (7). In particular from Eq. (6), after some calculations, it is possible to show that time dependent functions $\beta_1(t)$ and $\mathcal{Z}_1(t)$ satisfy the following equations:

$$\frac{1}{2\beta_1}\frac{d\beta_1}{dt} = -2\nu_1\mathcal{D}_1(t)\beta_1\mathcal{Z}_1^{q_1-1} + k_1(t)\,, \tag{17}$$

$$-\frac{1}{\mathcal{Z}_1}\frac{d\mathcal{Z}_1}{dt} = -2\nu_1\mathcal{D}_1(t)\beta_1\mathcal{Z}_1^{q_1-1} + k_1(t)\,. \tag{18}$$

For the other time dependent functions, i.e., $\beta_2(t)$ and $\mathcal{Z}_2(t)$, we have that

$$\frac{1}{2\beta_2}\frac{d\beta_2}{dt} = -2\nu_2\mathcal{D}_2(t)\beta_2\mathcal{Z}_2^{q-1} + k_2(t)\,, \tag{19}$$

$$-\frac{1}{\mathcal{Z}_2}\frac{d\mathcal{Z}_2}{dt} = -2\nu_2\mathcal{D}_2(t)\beta_2\mathcal{Z}_2^{q-1} + k_2(t)\,. \tag{20}$$

Note that these equations for $\beta_1(t)$, $\mathcal{Z}_1(t)$, $\beta_2(t)$, and $\mathcal{Z}_2(t)$ are coupled by the diffusion coefficients given by Eqs. (3) and (4), which can be written as follows:

$$\mathcal{D}_1(t) = D_1\mathcal{I}_2/\left(\mathcal{Z}_2^{\nu}(t)\sqrt{\beta_2(t)}\right) \text{ and}$$
$$\mathcal{D}_2(t) = D_2\mathcal{I}_1/\left(\mathcal{Z}_1^{\nu}(t)\sqrt{\beta_1(t)}\right), \tag{21}$$

with $\mathcal{I}_{1(2)} = \int_{-\infty}^{\infty} dx \exp_{q_{1(2)}}^{\nu_{1(2)}}(-x^2)$ and $0 < q_{1(2)} < 2$, i.e.,

$$\mathcal{I}_{1(2)} = \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{2-q_{1(2)}}{1-q_{1(2)}}\right)}{\sqrt{1-q_{1(2)}}\,\Gamma\left(\frac{3}{2}+\frac{2-q_{1(2)}}{1-q_{1(2)}}\right)} & \text{for } 0 < q_{1(2)} < 1\,, \\ \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2-q_{1(2)}}{q_{1(2)}-1}-\frac{1}{2}\right)}{\sqrt{q_{1(2)}-1}\,\Gamma\left(\frac{2-q_{1(2)}}{q_{1(2)}-1}\right)} & \text{for } 1 < q_{1(2)} < 2\,. \end{cases} \tag{22}$$

By substituting these equations in the previous set of equations for the time dependent functions, we may simplify them in order to obtain the following set of equations:

$$\frac{1}{2\beta_1}\frac{d\beta_1}{dt} = -2\nu_1 D_1\mathcal{I}_2\beta_1\left(\frac{\tilde{\mathcal{I}}_1}{\beta_1}\right)^{q_1-1}\left(\frac{\tilde{\mathcal{I}}_2}{\beta_2}\right)^{q_2-1} + k_1(t), \tag{23}$$

$$\frac{1}{2\beta_2}\frac{d\beta_2}{dt} = -2\nu_2 D_2\mathcal{I}_1\beta_2\left(\frac{\tilde{\mathcal{I}}_1}{\beta_1}\right)^{q_1-1}\left(\frac{\tilde{\mathcal{I}}_2}{\beta_2}\right)^{q_2-1} + k_2(t)\,. \tag{24}$$

In order to obtain the previous set of equations, we also use the normalization condition of $\rho_1(x_1,t)$ and $\rho_2(x_2,t)$, i.e., $\int_{-\infty}^{\infty} dx_{1(2)}\rho_{1(2)}(x_{1(2)},t) = 1$, which implies in $\mathcal{Z}_1\sqrt{\beta_1} = \tilde{\mathcal{I}}_1$ and $\mathcal{Z}_2\sqrt{\beta_2} = \tilde{\mathcal{I}}_2$ with

$$\tilde{\mathcal{I}}_{1(2)} = \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{1-q_{1(2)}}\right)}{\sqrt{1-q_{1(2)}}\,\Gamma\left(\frac{3}{2}+\frac{1}{1-q_{1(2)}}\right)} & \text{for } 0 < q_{1(2)} < 1\,, \\ \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{q_{1(2)}-1}-\frac{1}{2}\right)}{\sqrt{q_{1(2)}-1}\,\Gamma\left(\frac{1}{q_{1(2)}-1}\right)} & \text{for } 1 < q_{1(2)} < 2\,. \end{cases} \tag{25}$$

Thus, the time functions $\beta_1(t)$ and $\beta_2(t)$ can be obtained by solving the Eqs. (23) and (24). It is worthing mentioning that the behavior of $1/\beta_1(t)$ and $1/\beta_2(t)$ are directly related to the mean square displacement of the distributions related to these time dependent functions. This feature can be verified by performing some calculations in order to show that $\sigma^2_{1(2)} = \langle(x-\langle x\rangle)^2\rangle_{1(2)} \propto 1/\beta_{1(2)}(t)$.

Fig. 1 illustrates the behavior of $1/\beta_1(t)$ and $1/\beta_2(t)$ for different scenarios obtained by numerically solving the previous set of equations for $\beta_1(t)$ and $\beta_2(t)$. In particular, Fig. 1a shows that the coupling between the equations by the diffusive term, for $q_1 = q_2 = q$ with $q > 1$, results in a subdiffusive behavior for the species 2, instead of a stationary situation. In Fig. 1b, we observe that for $q_1 = q_2$ with $q < 1$, the effect of 1 on 2 for long times does not produce a dispersion, but a freezing process which results in a trapping of system 2 at a region. This feature is evidenced in Fig. 5 by illustrating the behavior of $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$ related to the diffusive term. In particular, it is possible to observe that, in the asymptotic limit of long times, $\mathcal{D}_2(t)$ decreases with time – see the Fig. 2a, the red dashed line, directly influencing the stochastic term present in Eq. (2) as shown in Fig. 2b. Fig. 3 considers $q_1 \neq q_2$, i.e., $q_1 = 1.1$ and $q_2 = 0.5$, with $k_1 = 0$ and $k_2 \neq 0$. It shows that system 1 modifies the stationary state of system 2 re-

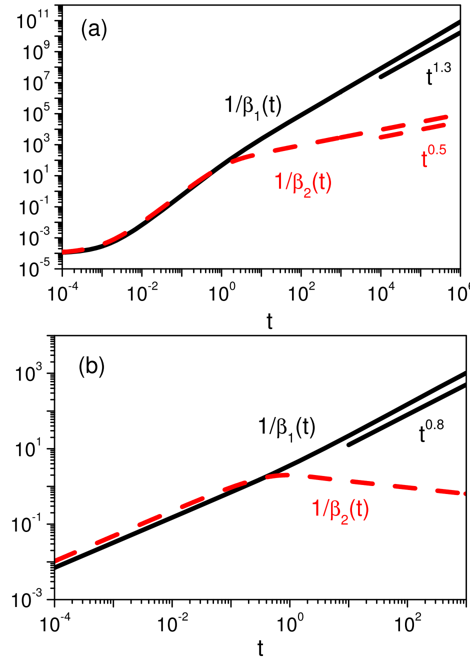

Fig. 1. This figure shows the behavior of $1/\beta_1(t)$ and $1/\beta_2(t)$ for $q_1 = q_2 = q = 1.5$ (Fig. 1a) and $q_1 = q_2 = q = 0.5$ (Fig. 1b) by considering that $k_1 = 0$ and $k_2 \neq 0$. We use, for simplicity, $D_1 = 1$, $D_2 = 1.5D_1$, and $k_2 = 1$, in arbitrary unities.

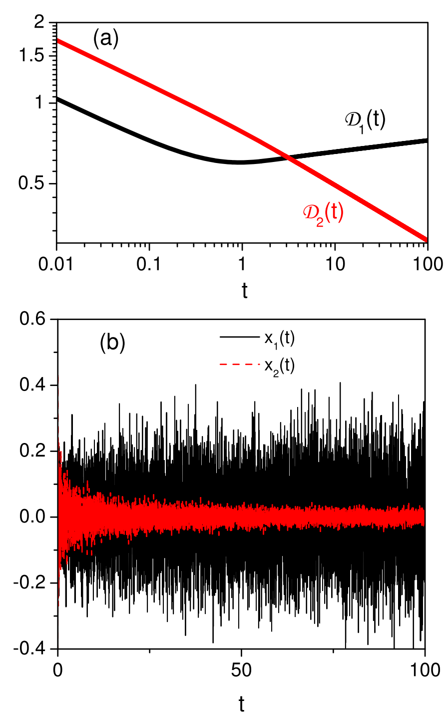

Fig. 2. Fig. 2a shows the behavior of $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$ for $q_1 = q_2 = q = 0.5$, with $k_1 = 0$ and $k_2 \neq 0$. Fig. 2b illustrates the behavior of $x_1(t)$ and $x_2(t)$ related to the $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$ presented in Fig. 2a. We use, for simplicity, $D_1 = 1$, $D_2 = 1.5D_1$, and $k_2 = 1$, in arbitrary unities. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

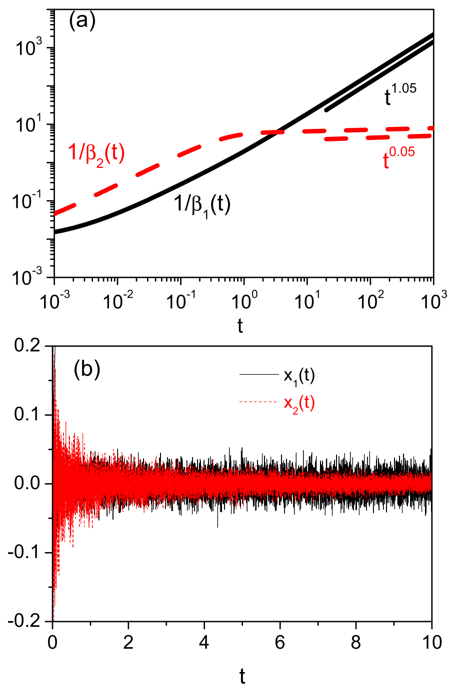

Fig. 3. Fig. 3a shows the behavior of $1/\beta_1(t)$ and $1/\beta_2(t)$ for $q_1 = 1.1$ and $q_2 = 0.5$, with $k_1 = 0$ and $k_2 \neq 0$. Fig. 3b illustrates the behavior of $x_1(t)$ and $x_2(t)$ related to the $q_1 = 1.1$ and $q_2 = 0.5$ presented in Fig. 3a. We use, for simplicity, $D_1 = 1$, $D_2 = 1.5D_1$, and $k_2 = 1$ in arbitrary unities. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

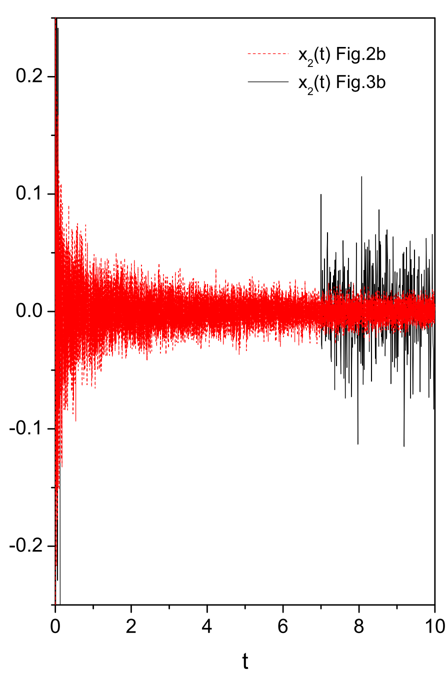

Fig. 4. This figure illustrates the behavior of the entropy of the system composed by 1 and 2 for $q_1 = q_2 = q = 0.5$ (Fig. 5a), with $k_1 = 0$ and $k_2 \neq 0$, by considering two different entropic additivities. In Fig. 5b, we show the time behavior of the entropy production related to the entropic additivity used in Fig. 5a, which shows that the standard entropic additivity leads us to an inconsistent result for the coupling between the nonlinear diffusion equation considered here. We use, for simplicity, $D_1 = 1$, $D_2 = 1.5D_1$, and $k_2 = 1$ in arbitrary unities. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

sulting in a subdiffusion. For this case, the solutions for Langevin equations are shown in Fig. 3b. In Fig. 4, we compare the solutions of the Langevin equation for the case $q_2 = 0.5$ when system 1 is considered under different conditions, i.e., $q_1 = 0.5$ and $q_1 = 1.1$. The effect on x_2 is remarkable, resulting in a freezing process when $q_1 = 0.5$ for system 1 and in a subdiffusion when $q_1 = 1.1$. Fig. 5a shows the behavior of the entropy of a system composed by 1 and 2 by considering two different entropic additivities. Fig. 5b shows that the standard entropic additivity leads to an inconsistent result for the entropy production. This feature evidences that the suitable statistics to analyze systems with this coupling is obtained with Tsallis entropy [22], as pointed out in Ref. [27].

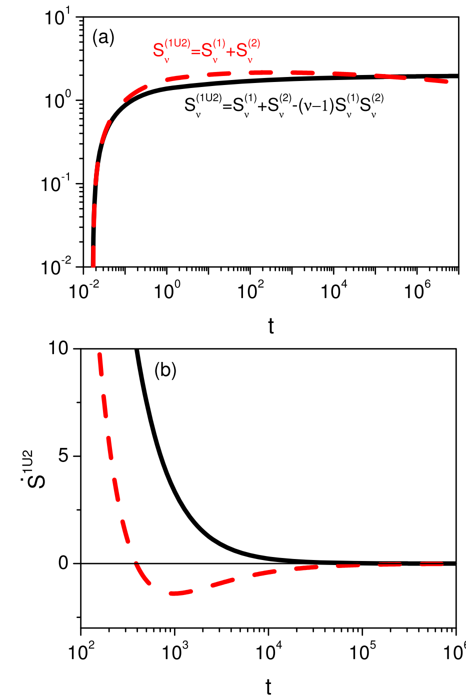

Fig. 5. This figure illustrates the behavior of the entropy of the system composed by 1 and 2 for $q_1 = q_2 = q = 0.5$ (Fig. 5a), with $k_1 = 0$ and $k_2 \neq 0$, by considering two different entropic additivities. In Fig. 5b, we show the time behavior of the entropy production related to the entropic additivity used in Fig. 5a, which shows that the standard entropic additivity leads us to an inconsistent result for the coupling between the nonlinear diffusion equation considered here. We use, for simplicity, $D_1 = 1$, $D_2 = 1.5D_1$, and $k_2 = 1$ in arbitrary unities.

An analytical solution for Eqs. (23) and (24) can be obtained by considering that the time dependent parameters satisfying the relation $D_1\mathcal{I}_2\nu_1\beta_1(t) = D_2\mathcal{I}_1\nu_2\beta_2(t)$ with $k_1(t) = k_2(t) = k(t)$. For this case, we can obtain $\beta_1(t)$ and, consequently, the other time dependent functions by using the previous relations. In this sense, Eq. (23) can be written as

$$\frac{1}{2\beta_1}\frac{d\beta_1}{dt} = -2\beta_1\nu_1 D_1\mathcal{I}_2\left(\frac{\bar{\mathcal{I}}_1}{\sqrt{\beta_1}}\right)^{q_1-1}\left(\sqrt{\frac{\nu_2\mathcal{D}_2\mathcal{I}_1}{\nu_1\mathcal{D}_1\mathcal{I}_2}}\frac{\bar{\mathcal{I}}_2}{\sqrt{\beta_1}}\right)^{q_2-1} + k(t) \tag{26}$$

whose solution, for an arbitrary initial condition $\beta_1(0)$, is given by

$$\frac{\beta_1(t)}{\beta_1(0)} = e^{2\int_0^t k(t')dt'}\left(1 + \mathcal{B}_1\int_0^t dt' e^{2\bar{\nu}\int_0^{t'} k(t'')dt''}\right)^{-\frac{1}{\bar{\nu}}}, \tag{27}$$

with

$$\mathcal{B}_1 = 2\nu_1 D_1\mathcal{I}_2\bar{\mathcal{I}}_1^{q_1-1}\left(\sqrt{\frac{\nu_2\mathcal{D}_2\mathcal{I}_1}{\nu_1\mathcal{D}_1\mathcal{I}_2}}\bar{\mathcal{I}}_2\right)^{q_2-1}\beta_1^{\bar{\nu}}(0) \tag{28}$$

and $\bar{\nu} = 2 - (q_1 + q_2)/2$. By using Eq. (27), it is possible to obtain $\mathcal{Z}_1(t)$, $\beta_2(t)$, and $\mathcal{Z}_2(t)$ by using the previous relations, as mentioned before. For instance, by using the normalization condition, i.e. $\mathcal{Z}_1\sqrt{\beta_1} = \bar{\mathcal{I}}_1$, it is possible to find $\mathcal{Z}_1(t)$ yielding

$$\frac{\mathcal{Z}_1(t)}{\mathcal{Z}_1(0)} = e^{\int_0^t k(t')dt'} \left(1 + \mathcal{B}_1 \int_0^t dt' e^{2\bar{\nu}\int_0^{t'} k(t'')dt''}\right)^{\frac{1}{2\bar{\nu}}} . \quad (29)$$

For the other time dependent functions, we have that

$$\frac{\beta_2(t)}{\beta_2(0)} = e^{2\int_0^t k(t')dt'} \left(1 + \mathcal{B}_2 \int_0^t dt' e^{2(2-q)\int_0^{t'} k(t'')dt''}\right)^{-\frac{1}{\bar{\nu}}} , \quad (30)$$

and

$$\frac{\mathcal{Z}_2(t)}{\mathcal{Z}_2(0)} = e^{\int_0^t k(t')dt'} \left(1 + \mathcal{B}_2 \int_0^t dt' e^{2(2-q)\int_0^{t'} k(t'')dt''}\right)^{\frac{1}{2\bar{\nu}}} , \quad (31)$$

with

$$\mathcal{B}_2 = 2\nu_2 D_2 \mathcal{I}_1 \bar{\mathcal{I}}_2^{q_2-1} \left(\sqrt{\frac{\nu_1 \mathcal{D}_1 \mathcal{I}_2}{\nu_2 \mathcal{D}_2 \mathcal{I}_2}} \bar{\mathcal{I}}_1\right)^{q_1-1} \beta_2^{\bar{\nu}}(0) . \quad (32)$$

By using these results, the case $k(t) = 0$ can be obtained. In particular, it results in

$$\frac{\beta_1(t)}{\beta_1(0)} = (1 + \mathcal{B}_1 t)^{-\frac{1}{\bar{\nu}}} \text{ and } \frac{\beta_2(t)}{\beta_2(0)} = (1 + \mathcal{B}_2 t)^{-\frac{1}{\bar{\nu}}} . \quad (33)$$

The results for $\beta_1(t)$ and $\beta_2(t)$ shows that for long times, i.e., $t >> \max\{1/\mathcal{B}_1, 1/\mathcal{B}_2\}$, the spreading on the distributions has the same time dependence, i.e., $\sigma_1^2 \propto \sigma_2^2 \propto t^{1/\bar{\nu}}$, where $\bar{\nu} > 1$ and $\bar{\nu} < 1$ correspond to the cases sub- and superdiffusive, respectively.

3. Conclusions

We have investigated the solutions for a set of coupled nonlinear Fokker–Planck equations. The coupling between the nonlinear Fokker–Planck equations is performed through the diffusive, which is related to the stochastic variable, i.e., random forces, responsible for the spreading of the system. By solving these equations, we have obtained stationary solutions and time-dependent solutions for a time-dependent external force with a linear spatial dependence. We have observed that, depending on the conditions, which are employed on the nonlinear Fokker–Planck equations, different diffusive behaviors can be obtained for each system, where their relaxations are modified by each other. This point is illustrated in Figs. 1, 2, and 3. In particular, Fig. 2 has shown that interaction of system 2 with system 1 may decrease the amplitude of the stochastic force leading to a freezing process when $q_1 = q_2 = 0.5$ or a subdiffusion for system 2 when $q_1 = 1.1$ and $q_2 = 0.5$. From the analytical results, we have verified that both systems, for long times, asymptotically have the same time dependence for the mean square displacement, i.e., $\sigma_1^2 \propto \sigma_2^2 \propto t^{1/\bar{\nu}}$. We have also investigated, from the point of view of thermostatistics, a suitable context to consider the system composed by systems 1 and 2. For this, we have considered the case presented in Fig. 2 and analyzed the entropy by considering two different entropic additivities. One of the cases has produced inconsistent results for the entropy production evidencing that the suitable entropic additivity is the one connected to the Tsallis entropy. Finally, we hope that the results presented here may be useful to investigate situations related to anomalous diffusion and nonlinear Fokker–Planck equations.

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