PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

CrossMark
click for updates

Cite this article: Lenzi EK, Ribeiro HV, Tateishi AA, Zola RS, Evangelista LR. 2016 Anomalous diffusion and transport in heterogeneous systems separated by a membrane. *Proc. R. Soc. A* **472**: 20160502. http://dx.doi.org/10.1098/rspa.2016.0502

Received: 20 June 2016
Accepted: 13 October 2016

Subject Areas:
statistical physics

Keywords:
anomalous diffusion, fractional diffusion equations, heterogeneous media, transport

Author for correspondence:
E. K. Lenzi
e-mail: eklenzi@uepg.br

Anomalous diffusion and transport in heterogeneous systems separated by a membrane

E. K. Lenzi[1], H. V. Ribeiro[2], A. A. Tateishi[3], R. S. Zola[2,4] and L. R. Evangelista[2]

[1]Departamento de Física, Universidade Estadual de Ponta Grossa, Ponta Grossa, Paraná 87030-900, Brazil
[2]Departamento de Física, Universidade Estadual de Maringá, Maringá, Paraná 87020-900, Brazil
[3]Departamento de Física, Universidade Tecnológica Federal do Paraná, Pato Branco, Paraná 85503-390, Brazil
[4]Departamento de Física, Universidade Tecnológica Federal do Paraná, Apucarana, Paraná 86812-460, Brazil

EKL, 0000-0003-3853-1790

Diffusion of particles in a heterogeneous system separated by a semipermeable membrane is investigated. The particle dynamics is governed by fractional diffusion equations in the bulk and by kinetic equations on the membrane, which characterizes an interface between two different media. The kinetic equations are solved by incorporating memory effects to account for anomalous diffusion and, consequently, non-Debye relaxations. A rich variety of behaviours for the particle distribution at the interface and in the bulk may be found, depending on the choice of characteristic times in the boundary conditions and on the fractional index of the modelling equations.

1. Introduction

Many of the fundamental problems in biology and engineering are characterized by systems separated by a membrane [1,2]. In a general context, one can emphasize that membranes mainly act as barriers and as transporters. In other words, they are responsible for (i) the selectivity of particles by means of sorption and desorption processes and, consequently, (ii) the

THE ROYAL SOCIETY
PUBLISHING

translocation of particles from one region to the other. In the complex and dynamical environment of living cells, the membrane is key to ensure the suitable transport of information and material between the cell's outer and inner regions [1], i.e. it is selectively permeable to organic molecules and ions [3]. The dynamics of these molecules and ions relies on transport processes that are crucial for maintaining life, including simple diffusion, facilitated diffusion, osmosis and active transport. In the simplest case, the diffusion rate is directly proportional to the concentration of the diffusing substance. For facilitated diffusion and active transport, the rate depends on the number of channels and carrier proteins in the membrane, and approaches its maximum value as the number of carriers become saturated [3,4]. Adding to the complex permeability of membranes is the fact that some proteins, called peripheral membrane proteins, adhere only temporarily to the membrane regulating cell signalling and then detaching from the membrane [5]. For engineering, the development of synthetic (or inorganic) membranes as permeable barriers are very suitable for separation applications and for reaction enhancement [6,7]. Compared with the organic membrane, the inorganic ones exhibit higher thermal and chemical stabilities [2]. For instance, important applications can be found in the water treatment, where the membrane plays an important role to the removal of unwanted chemicals and viruses from contaminated sources of water [8], making it possible to conduct environmentally important reactions without adding harmful and expensive chemicals [9].

The choice of appropriate boundary conditions is crucial to achieve a suitable description of these complex phenomena in connection with the specific processes undergone by particles in the vicinity of the membrane, which involve also translocation through it [10–12]. Boundary conditions obtained from the experimental scenario also determine the dynamics of particles in the bulk and may lead to an anomalous diffusion which, frequently, manifest different diffusive regimes [13]. One of the main characteristics of this anomalous diffusion is the nonlinear behaviour of the mean square displacement, i.e. $\langle (x - \langle x \rangle)^2 \rangle \propto t^{\alpha}$ ($\alpha < 1$ and $\alpha > 1$ correspond to subdiffusion and superdiffusion, respectively), which is, in general, related to a stochastic process with non-Markovian characteristics. It is worth mentioning that systems characterized by long tailed distributions, i.e. Lévy distributions, also exhibit anomalous diffusion. Several models have been proposed in order to explain this complicated behaviour from the physical point of view [14–21]. In most of the cases, stochastic models based on the diffusion equation and simulation methods are employed to face the problem [22].

Here, from the formal point of view, we investigate the diffusion in a heterogenous media separated by a membrane, whose processes are described in terms of kinetic equations, responsible for the selectivity of particles by means of a sorption and desorption process and translocation. In our model, both, kinetic and boundary conditions, were modified in order to embody unusual relaxations and possible anomalous diffusion effects, by incorporating convolutions between the reaction rates and densities, leading to non-Debye relaxation processes, i.e. memory effects [23,24]. We also consider that the particle diffusion in the bulk is governed by fractional diffusion equations [25]. In this manner, the problem formulated here is general enough to incorporate several relevant particular cases and enable us to explore different scenarios.

2. The problem: diffusion and kinetics

Let us consider an infinity medium where all the relevant quantities diffuse in one dimension (the x-direction). For simplicity, we consider that there is a membrane located at $x = 0$, so it divides the space in two regions, $x > 0$ (side 1) and $x < 0$ (side 2). The membrane's thickness ξ is sufficiently thin to assure that only the kinetic processes are relevant in this region (figure 1). We consider that the distributions of the particles in the bulk are governed by the fractional diffusion equations

$$\frac{\partial}{\partial t}\rho_1(x,t) = \mathcal{K}_1\, {}_0\mathcal{D}_t^{1-\alpha_1}\left[\frac{\partial^2}{\partial x^2}\rho_1(x,t)\right], \qquad (2.1)$$

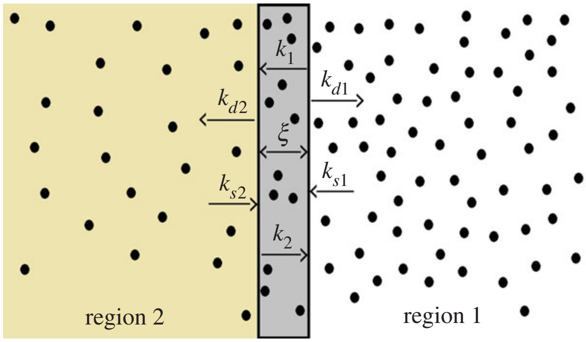

Figure 1. The steps of the formalism proposed here may be summarized as follows: the particles present in region 1 are sorbed (with a rate k_{s1}) by a membrane placed between two media and, by means of a kinetic process, the substance may be transported from region 1 to region 2, with a rate k_1. Likewise, the substance in region 2 may also be sorbed (with a rate k_{s2}) by the surface and then transported to region 1, with a rate k_2. Note that the membrane thickness ξ is considered sufficiently small, i.e. $|\xi| \to 0$, to avoid the diffusion in this region. (Online version in colour.)

for $0 < x < \infty$, and

$$\frac{\partial}{\partial t}\rho_2(x,t) = \mathcal{K}_2\, {}_0\mathcal{D}_t^{1-\alpha_2}\left[\frac{\partial^2}{\partial x^2}\rho_2(x,t)\right], \tag{2.2}$$

for $-\infty < x < 0$. In equations (2.1) and (2.2), $\mathcal{K}_1$ and $\mathcal{K}_2$ are, respectively, the generalized diffusion coefficients of regions 1 and 2. The quantities ρ_1 and ρ_2 represent the density of particles in each side and the fractional time derivative present in equations (2.1) and (2.2) is the Riemann–Liouville one, i.e.

$${}_0\mathcal{D}_t^{1-\alpha_{1(2)}}[\rho_{1(2)}(x,t)] = \frac{1}{\Gamma(\alpha_{1(2)})}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{d}t'\,\frac{\rho_{1(2)}(x,t')}{(t-t')^{1-\alpha_{1(2)}}}, \tag{2.3}$$

with $0 < \alpha_{1(2)} < 1$ [26,27]. Note that the values $0 < \alpha_1 < 1$ and $0 < \alpha_2 < 1$ are connected with a subdiffusive process, whereas $\alpha_1 = \alpha_2 = 1$ corresponds to usual diffusion. It is worth noting that the fractional diffusion equation is an important tool for describing anomalous diffusion in general (e.g. [28–30]). On each side of the membrane, the following equations apply:

$$\mathcal{K}_1\, {}_0\mathcal{D}_t^{1-\alpha_1}\left[\frac{\partial}{\partial x}\rho_1(x,t)\right]\Bigg|_{x=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{\text{side},1}(t) + \int_0^t k_1(t-t')\rho_1(0,t')\,\mathrm{d}t' - \int_0^t k_2(t-t')\rho_2(0,t')\,\mathrm{d}t' \tag{2.4}$$

and

$$\mathcal{K}_2\, {}_0\mathcal{D}_t^{1-\alpha_2}\left[\frac{\partial}{\partial x}\rho_2(x,t)\right]\Bigg|_{x=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{\text{side},2}(t) - \int_0^t k_2(t-t')\rho_2(0,t')\,\mathrm{d}t' + \int_0^t k_1(t-t')\rho_1(0,t')\,\mathrm{d}t', \tag{2.5}$$

where $k_1(t)$ and $k_2(t)$ are related to the rate of particle transport through the membrane from the one side to the other side. This set of equations, which connect the current density from both sides, are intended to describe the sorption, desorption and transport of the particles from one region (e.g. region 1) to the other (e.g. region 2). In these equations, $\mathcal{C}_{\text{side},1}(t)$ and $\mathcal{C}_{\text{side},2}(t)$ represent the density of particles at the membrane on each side, obtained from the bulk by a sorption process. The first term in the right side of equations (2.4) and (2.5) gives the time variation of the particles sorbed by the surfaces of the membrane and the others terms are related to the transport of the particles sorbed from the one side to the other side. For the sorption process and desorption processes on the surface, we propose that they are modelled by the following kinetic equations [13]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{\text{side},1}(t) = k_{s1}\rho_1(0,t) - \int_0^t k_{d1}(t-t')\mathcal{C}_{\text{side},1}(t')\,\mathrm{d}t' \tag{2.6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}_{\mathrm{side},2}(t) = k_{s2}\rho_2(0,t) - \int_0^t k_{d2}(t-t')\mathcal{C}_{\mathrm{side},2}(t')\,\mathrm{d}t'. \tag{2.7}$$

In the above equations, $\rho_1(0,t)$ and $\rho_2(0,t)$ are the bulk density just in front of the membrane, on sides 1 and 2, respectively. k_{si} $(i = 1, 2)$ are parameters connected to the sorption phenomena, being related to a characteristic sorption time $\tau_i \propto 1/k_{si}$, and $k_{di}(t)$ is a kernel that governs the desorption phenomena. These equations state that the time variation of the surface density of sorbed particles at a given side depends on the bulk density of particles just in front of the membrane, and on the surface density of particles already sorbed [13]. They extend the usual kinetic equations (Langmuir approximation) to situations characterized by non-usual relaxations, i.e. non-Debye relaxations for which a non-exponential behaviour of the densities can be obtained, depending on the choice of the kernels [13,31]. The underlying physical motivation of the time-dependent rate coefficients can be related with the fractal nature, low-dimensionality or macromolecular crowding of the medium [32–35], and even with the anomalous molecular diffusion [36]. Moreover, a linear response approach [37] was proposed to describe this temporal behaviour and there is a possible connection with the generalization of the mass action law [38,39]. From a phenomenological point of view, the choice of the kernels of the equations (2.6) and (2.7) can be related to surface irregularities [32], which are important in adsorption–desorption, to diffusion, to catalysis processes, and to microscopic parameters representing the van der Waals interaction between the particles and the surfaces [40]. A kernel like $k_{di}(t)$ has been used in several contexts to express non-Debye relaxation, yielding non-trivial behaviour description and allowing for different or combined effects in a single kinetic equation [13].

In a few words, equations (2.6) and (2.7) are proposed to represent the sorption of the particles on each surface with the rates k_{s1} and k_{s2}. After the sorption process, the particles present on the surface can be desorbed, with the rates $k_{d1}(t)$ and $k_{d2}(t)$. They can also be transported throughout of the membrane with the rates $k_1(t)$ and $k_2(t)$. Thus, the usual current balance [13,31] is modified by the second and third terms on the right side of equations (2.4) and (2.5) in order to represent the transport of particles from side 1 to side 2 and vice versa. For a cell membrane, $k_1(t)$ and $k_2(t)$ may be related to a continuous-time discrete Markov process of two stages for the transition rates as the channel can exist either in a closed or in an open state on each side. In this sense, they may be connected to the probability of opening the channel in a given side in order to make possible the transport of particles from the one side to the other. Furthermore, equations (2.4) and (2.5) can also be related to generalized reactive boundary conditions and the presence of fractional time derivatives are connected to the possibility of non-usual relaxation in the bulk, as discussed below. Formal derivations and possible applications in biological contexts of the fractional diffusion equation with reactive boundary conditions can be found in [41–43]. In particular, due to the non-Markovian nature of subdiffusion, these conditions seem to be more suitable than the usual reaction–diffusion equation with reaction term independent on the transport one [41]. It is also interesting to mention that equations (2.4) and (2.5) are coupled with equations (2.6) and (2.7) in such way that processes occurring in one side modifies the dynamic of the other side. The searched solutions are also subjected to the homogeneous boundary conditions $\partial_x\rho_1(\infty,t) = 0$ and $\partial_x\rho_2(-\infty,t) = 0$ to conserve the number of particles present in the system. For a non-conservative situation, it is possible to consider inhomogeneous boundary conditions which may be related to a source or sink.

The boundary conditions incorporated in the diffusion equations imply that

$$\mathcal{C}_{\mathrm{side},1}(t) + \mathcal{C}_{\mathrm{side},2}(t) + \int_{-\infty}^{0} \rho_2(x,t)\,\mathrm{d}x + \int_0^{\infty} \rho_1(x,t)\,\mathrm{d}x = \mathrm{const.}, \tag{2.8}$$

which is a direct consequence of the conservation of the total number of particles present in the system. Equation (2.8) can be obtained by performing an integration in equations (2.1) and (2.2) which, after using equations (2.4) and (2.5), become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \rho_1(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},1}(t)\right) = \int_0^t k_2(t-t')\rho_2(0,t')\,\mathrm{d}t' - \int_0^t k_1(t-t')\rho_1(0,t')\,\mathrm{d}t' \tag{2.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^0 \rho_2(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},2}(t)\right) = \int_0^t k_1(t-t')\rho_1(0,t')\,\mathrm{d}t' - \int_0^t k_2(t-t')\rho_2(0,t')\,\mathrm{d}t'. \tag{2.10}$$

By adding equations (2.9) and (2.10), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \rho_1(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},1}(t)\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^0 \rho_2(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},2}(t)\right), \tag{2.11}$$

from which equation (2.8) can be obtained. Equation (2.11) also shows that the mass (number of particles) variation on side 1 is connected to the variations on side 2. In particular, the negative sign shows that the variation of particles on one side (gain or loss) produces an opposite variation on the other side. The set of equations (2.6)–(2.8) covers a general scenario from which some specific situations can be investigated.

Let us consider some concrete cases in order to illustrate how previous formalism works. For this, we initially suppose that all the particles are located in the region $x > 0$ (region 1), such that the initial conditions are, in a general form, $\rho_1(x,0) = \varphi(x)$ (with $\int_0^\infty \mathrm{d}x\rho_1(x,0) = 1$), $\rho_2(x,0) = 0$ and $\mathcal{C}_{\mathrm{side},1}(0) = \mathcal{C}_{\mathrm{side},2}(0) = 0$). As time evolves, region 2 gets occupied by particles from region 1 as a product of the membrane transport and, consequently, the density of particles at region 2 is totally dependent on the role developed by the membrane. The general solution in different media may be established by using the Laplace transform, where $\rho(x,s) = \mathcal{L}\{\rho(x,t)\}[\mathcal{L}\{\ldots\} = \int_0^\infty(\ldots)\mathrm{e}^{-st}\,\mathrm{d}t]$, and the Green function approach. We note that the solution of equation (2.2), in the Laplace space, is given by

$$\rho_2(x,s) = \omega_2(s)\mathrm{e}^{-\sqrt{s^{\alpha_2}/\mathcal{K}_2}|x|} \tag{2.12}$$

and

$$\omega_2(s) = \frac{(s+k_{d2}(s))k_1(s)}{(s\sqrt{\mathcal{K}_2/s^{\alpha_2}} + k_2(s))(s+k_{d2}(s)) + sk_{s2}}\rho_1(0,s), \tag{2.13}$$

from which it is possible to show that

$$\mathcal{C}_{\mathrm{side},2}(s) = \frac{(s+k_{d1}(s))k_{s2}k_1(s)}{(s\sqrt{\mathcal{K}_2/s^{\alpha_2}} + k_2(s))(s+k_{d2}(s))k_{s1} + sk_{s2}k_{s1}}\mathcal{C}_{\mathrm{side},1}(s), \tag{2.14}$$

which enables us to reduce equation (2.4) to

$$\mathcal{K}_1 s^{1-\alpha_1}\left[\frac{\partial}{\partial x}\rho_1(x,s)\right]\Big|_{x=0} = \omega_1(s)\rho_1(0,s), \tag{2.15}$$

with

$$\omega_1(s) = \frac{s}{s+k_{d1}(s)}k_{s1} + k_1(s) - \frac{(s+k_{d2}(s))k_2(s)k_1(s)}{(s\sqrt{\mathcal{K}_2/s^{\alpha_2}} + k_2(s))(s+k_{d2}(s)) + sk_{s2}}. \tag{2.16}$$

By means of the Green function approach, the solution for $\rho_1(x,s)$ can be written as

$$\rho_1(x,s) = -\int_0^\infty \mathcal{G}_1(x,x';s)\varphi(x')\,\mathrm{d}x', \tag{2.17}$$

in which the Green function is defined as

$$\begin{aligned}\mathcal{G}_1(x,x';s) = &-\frac{1}{2s\sqrt{\mathcal{K}_1/s^{\alpha_1}}}(\mathrm{e}^{-\sqrt{s^{\alpha_1}/\mathcal{K}_1}|x-x'|} + \mathrm{e}^{-\sqrt{s^{\alpha_1}/\mathcal{K}_1}|x+x'|})\\ &+ \frac{2\omega_1(s)}{s\sqrt{\mathcal{K}_1/s^{\alpha_1}} + \omega_1(s)}\frac{1}{2s\sqrt{\mathcal{K}_1/s^{\alpha_1}}}\mathrm{e}^{-\sqrt{s^{\alpha_1}/\mathcal{K}_1}|x+x'|}.\end{aligned} \tag{2.18}$$

rspa.royalsocietypublishing.org Proc. R. Soc. A 472: 20160502

Equation (2.12) can be rewritten, by using the previous results, as follows:

$$\rho_2(x,s) = \frac{(s+k_{d2}(s))k_1(s)}{(s\sqrt{\mathcal{K}_2/s^{\alpha_2}}+k_2(s))(s+k_{d2}(s))+sk_{s2}}\,\frac{1}{s\sqrt{\mathcal{K}_1/s^{\alpha_1}}+\omega_1(s)}\mathrm{e}^{-\sqrt{s^{\alpha_1}/\mathcal{K}_1}|x'|}\mathrm{e}^{-\sqrt{s^{\alpha_2}/\mathcal{K}_2}|x|}. \tag{2.19}$$

By analysing equation (2.18), we note that the processes present at the membrane have direct influence on the dynamics of the particles in region 1. The presence of $k_2(s)$, $k_{d2}(s)$ and k_{s2} in $\omega(s)$ implies that the sorption and desorption processes in region 2 and, consequently, the particle transport (from region 2 to region 1) modify the dynamic of the particles in region 1. After performing the inverse Laplace transform with $k_1(s) = k_1 = \text{const.}$, $k_{d1}(s) = k_{d1} = \text{const.}$, $k_2(s) = k_2 = \text{const.}$ and $k_{d2}(s) = k_{d2} = \text{const.}$, equations (2.18) and (2.12) can be written, respectively, as

$$\begin{aligned}\mathcal{G}_1(x,x';t) = &-\frac{1}{\sqrt{4\mathcal{K}_1 t^{\alpha_1}}}\left(\mathrm{H}^{1,0}_{1,1}\left[\frac{|x-x'|}{\sqrt{\mathcal{K}_1 t^{\alpha_1}}}\Bigg|^{(1-\alpha_1/2,\alpha_1/2)}_{(0,1)}\right] + \mathrm{H}^{1,0}_{1,1}\left[\frac{|x+x'|}{\sqrt{\mathcal{K}_1 t^{\alpha_1}}}\Bigg|^{(1-\alpha_1/2,\alpha_1/2)}_{(0,1)}\right]\right)\\ &+\int_0^t \frac{\mathrm{d}t'}{\sqrt{\mathcal{K}_1 t'^{\alpha_1}}}\varPi(t-t')\,\mathrm{H}^{1,0}_{1,1}\left[\frac{|x+x'|}{\sqrt{\mathcal{K}_1 t'^{\alpha_1}}}\Bigg|^{(1-\alpha_1/2,\alpha_1/2)}_{(0,1)}\right]\end{aligned} \tag{2.20}$$

and

$$\rho_2(x,t) = \int_0^t \frac{\mathrm{d}t'}{\sqrt{4\mathcal{K}_1 t'^{\alpha_2}}}\varUpsilon(t-t')\,\mathrm{H}^{1,0}_{1,1}\left[\frac{|x|}{\sqrt{\mathcal{K}_2 t'^{\alpha_2}}}\Bigg|^{(1-\alpha_2/2,\alpha_2/2)}_{(0,1)}\right], \tag{2.21}$$

with

$$\varUpsilon(t) = \varPhi(t) + \sum_{n=1}^{\infty}(-k_{s2})^n\int_0^t \mathrm{d}t_n \varXi(t_n - t_{n-1})\int_0^{t_n}\mathrm{d}t_{n-1}\varXi(t_{n-1}-t_{n-2})\cdots\int_0^{t_2}\mathrm{d}t_1\varXi(t_2-t_1)\varPhi(t_1), \tag{2.22}$$

where

$$\varXi(t) = \frac{1}{\sqrt{\mathcal{K}_2 t^{\alpha_2}}}E_{\alpha',\beta'}\left(-\frac{k_2}{\sqrt{\mathcal{K}_2}}t^{\alpha'}\right) - k_{d2}\mathrm{e}^{-k_{d2}t'}\int_0^t \mathrm{d}t'\frac{\mathrm{e}^{-k_{d2}t'}}{\sqrt{\mathcal{K}_2 t'^{\alpha_2}}}E_{\alpha',\beta'}\left(-\frac{k_2}{\sqrt{\mathcal{K}_2}}t^{\alpha'}\right) \tag{2.23}$$

and

$$\varPhi(t) = k_1\rho_1(0,t) - k_1\int_0^t \mathrm{d}t'\frac{k_2\rho_1(0,t')}{\sqrt{\mathcal{K}_2(t-t')^{\alpha_2}}}E_{\alpha',\beta'}\left(-\frac{k_2}{\sqrt{\mathcal{K}_2}}(t-t')^{\alpha'}\right) \tag{2.24}$$

and $\alpha' = \beta' = 1-\alpha_2/2$. The quantities $\varPi(t)$ and $I_{\omega_\varsigma}(t)$ introduced above are defined, respectively, as

$$\varPi(t) = I_{\omega_1}(t) + \sum_{n=1}^{\infty}(-1)^n\int_0^t \mathrm{d}t_n I_{\omega_1}(t)\cdots\int_0^{t_2}\mathrm{d}t_1 I_{\omega_1}(t_2-t_1)I_{\omega_1}(t_1) \tag{2.25}$$

and

$$I_{\omega_1}(t) = \frac{1}{\sqrt{\mathcal{K}_1}\varGamma(1+\alpha_1/2)}\int_0^t \mathrm{d}t'\frac{\omega_1(t')}{(t-t')^{\alpha_1/2}}. \tag{2.26}$$

In previous equations, $\mathrm{H}^{m,n}_{p,q}[x|^{(a_p,A_p)}_{(b_q,B_q)}]$ is the Fox H function [27] and $E_{\alpha,\beta}(x)$ is the two parameters Mittag–Leffler function—functions connected to anomalous relaxation of the system due to the presence of fractional time derivative in equations (2.1) and (2.2).

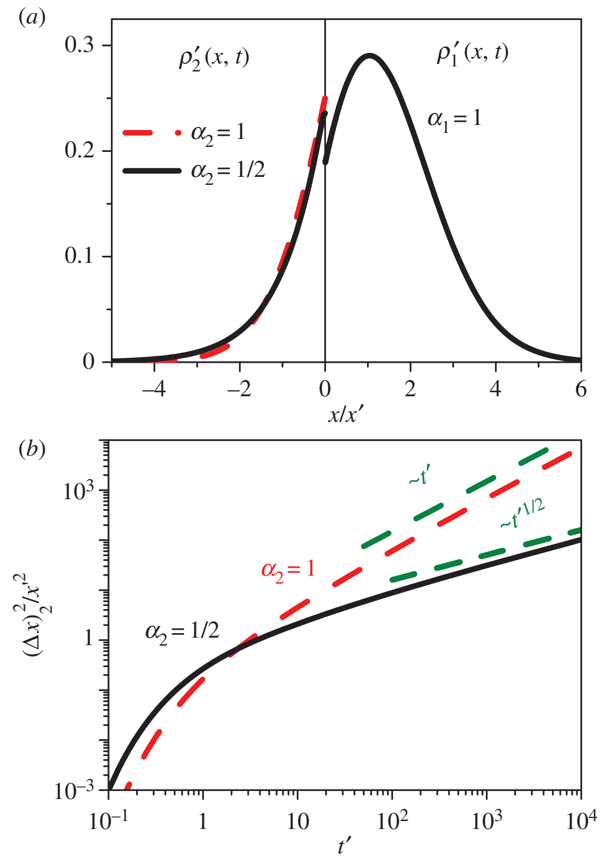

Figure 2. (*a*) Spatial profiles of the distributions $\rho_1'(x,t)$ and $\rho_2'(x,t)$ (where $\rho_1'(x,t) = x'\rho_1(x,t)$ and $\rho_2'(x,t) = x'\rho_2(x,t)$) for two different values of α_2, when $t = \tau_1$ (where $\tau_1 = (x'^2/\mathcal{K}_1)^{1/\alpha_1}$). (*b*) Behaviour of the mean square displacement ($(\Delta_2 x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ where the subindex 2 denotes region 2) for the particles present in region 2 with $t' = t/\tau_1$. In figure 2*b*, the dashed lines are guides to the eyes representing the diffusive regimes. We consider, for simplicity, $k_1(t) = \bar{k}_1\delta(t)$ (where $\bar{k}_1 = x'/\tau_1$), $k_{1d}(t) = k_{2d}(t) = 0$, $\tau_1 = \tau_2$ (where $\tau_2 = (x'^2/\mathcal{K}_2)^{1/\alpha_2}$), $k_{1s} = k_{2s} = 0$, and $k_2 = 0$, in arbitrary units. (Online version in colour.)

Now, let us first apply the previous results to the situation characterized by the particles being transported (e.g. osmotic process) from one side to the other, i.e. $k_2 = 0$ with $k_1 \neq 0$. In connection to a cell membrane, the constant k_1 may also be related to the opening channels rate, which are able to conduct the substance from one side to the other, as mentioned before. In this case, equations (2.9) and (2.10) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \rho_1(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},1}(t)\right) = -k_1\rho_1(0,t) \tag{2.27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^0 \rho_2(x,t)\,\mathrm{d}x + \mathcal{C}_{\mathrm{side},2}(t)\right) = k_1\rho_1(0,t). \tag{2.28}$$

These equations show that in region 1 the substance (number of particles) is decreasing and in region 2 the substance (number of particles) is increasing due to the transport across the membrane with the rate k_1.

Figure 2 shows the behaviour of the distribution for two different values of α_2 and the mean square displacement in region 2. In particular, we consider that $\alpha_1 = 1$ in region 1, and in region 2,

rspa.royalsocietypublishing.org Proc. R. Soc. A 472: 20160502

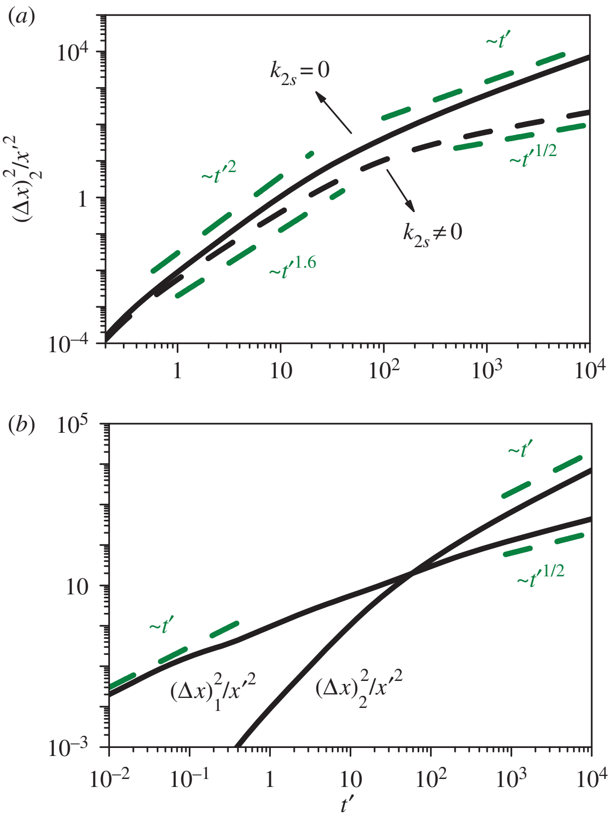

Figure 3. (*a*) Mean square displacement in region 2 for cases $k_{2s} = 0$ and $k_{2s} \neq 0$. These cases passes through a transient characterized by different regimes until them reach a critical concentration (region 2) and the behaviour becomes usual for $k_{2s} = 0$ or subdiffusive for $k_{2s} \neq 0$. (*b*) Behaviour of the mean square displacement for region 1 and in region 2, with $k_{2s} = 0$. Note that, in region 1, the regime is initially usual but, after some time has elapsed, it becomes subdiffusive as the membrane promotes adsorption, desorption and transport of particles, thus inducing particle crowding in region 1. The dashed lines are guides to the eyes representing the diffusive regimes. We consider, for simplicity, $k_{1s} = 10^2 x'/\tau_1$, $k_1(t) = \bar{k}_1 \delta(t)$ (where $\bar{k}_1 = x'/\tau_1$), $t' = t/\tau_1$, $k_{1d}(t) = \bar{k}_{1d}\delta(t)$ (where $\bar{k}_{1d} = 2/\tau_1$), $k_{2d}(t) = 0$, $\alpha_1 = \alpha_2 = 1$, $\tau_1 = \tau_2$ (where $\tau_1 = (x'^2/\mathcal{K}_1)^{1/\alpha_1}$ and $\tau_2 = (x'^2/\mathcal{K}_2)^{1/\alpha_2}$), $k_{2s} = x'/\tau_2$, and $k_2(t) = 0$, in arbitrary units. (Online version in colour.)

two values for α_2 are allowed: $\alpha_2 = 1$ and $\alpha_2 = \frac{1}{2}$. From figure 2*b*, we observe that the behaviour of the mean square displacement is asymptotically governed by the bulk equations for $k_{2s} = 0$ and may exhibit an anomalous behaviour depending on the α_2 value. For this case, in the asymptotic limit of long times, i.e. $t \to \infty$, we have $\langle (x - \langle x \rangle)^2 \rangle \to t^{\alpha_2}$ showing that bulk effects have a pronounced role on this diffusive regime. The initial behaviour manifested in figure 2*b* by the particles in region 2 is an evidence of the confinement of particles by the membrane during the process of passage from region 1 to 2. For intermediate times, we have the presence of different regimes, e.g. as evidenced in figure 2*b* for $\alpha_2 = 1$ by the straight lines. Similar feature also occurs for $\alpha_2 = \frac{1}{2}$ until the system reaches the asymptotic regime characterized by subdiffusion. The different regimes manifested for both cases, presented in figure 2*b*, show the influence of the membrane which represents an obstacle to the passage of the particles from the one side to the other side. They are transitory until the movement of the system achieves the regime governed by the bulk equation for long times. In fact, for this case, in the asymptotic limit of long times, i.e. $t \to \infty$, we have $\langle (\Delta_2 x)^2 \rangle \to t^{\alpha_2}$ showing that bulk effects have a pronounced role on this diffusive regime.

The spreading of the system also depends on the processes at the surface on region 2, which may involve different diffusion regimes. This feature is illustrated in figure 3*a* by comparing the

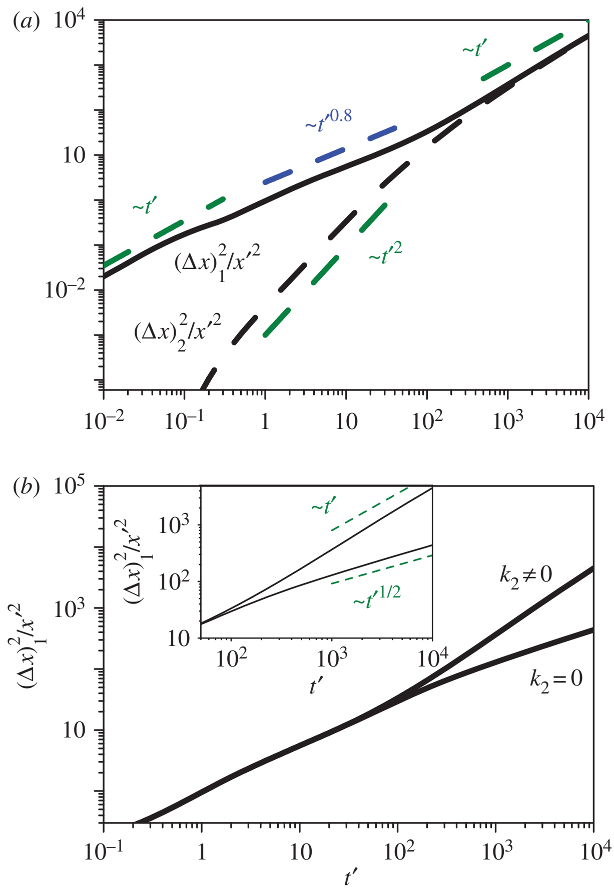

Figure 4. (*a*) Mean square displacement in regions 1 and 2 with $k_2 \neq 0$ and $k_{2s} \neq 0$. Region 1 starts with usual diffusion and passes through different regimes, and for long times recovers the usual behaviour. In particular, this asymptotic behaviour in region 1 is due to the reverse transport of particles, i.e. $k_2 \neq 0$, from region 2 to region 1, as shown in (*b*) when compared with case $k_2 = 0$. In the inset of figure 4*b*, we show that $k_2 \neq 0$ led us asymptotically to a usual diffusion and $k_2 = 0$ a subdiffusive regime. Region 2, similarly to the previous case, manifests a different behaviour followed by a usual diffusion. We consider, for simplicity, $k_{1s} = 10^2 x'/\tau_1$, $k_1 = \bar{k}_1\delta(t)$ (where $\bar{k}_1 = x'/\tau_1$), $k_{1d} = (2/\tau_1)\delta(t)$, $t' = t/\tau_1$, $\tau_1 = \tau_2$ (where $\tau_1 = (x'^2/\mathcal{K}_1)^{1/\alpha_1}$ and $\tau_2 = (x'^2/\mathcal{K}_2)^{1/\alpha_2}$), $\alpha_1 = \alpha_2 = 1$, $k_{2d}(t) = [1/(2\tau_2)]\delta(t)$, $k_{2s} = x'/\tau_2$, and $k_2(t) = \bar{k}_2\delta(t)$ (where $\bar{k}_2 = x'/\tau_2$), in arbitrary units. (Online version in colour.)

cases $k_{s2} = 0$ (solid black line) and $k_{s2} \neq 0$ (dashed black line) with $k_2 = 0$. We verify that initially the behaviour of the particles transported from region 1 to region 2 after leaving the membrane is superdiffusive (green dashed line) followed by an usual behaviour for $k_{s2} = 0$ or a subdiffusive one for $k_{s2} \neq 0$. In figure 3*b*, we show that in region 1, which initially presents usual diffusion, the process evolves towards anomalous diffusion for long times. This fact can be attributed to the surface, which is sorbing, desorbing and transporting particles out of region 1.

Figure 4*a* shows the behaviour of the mean square displacement for both regions when case $k_2 \neq 0$ is considered, i.e. the membrane also permits a reverse (e.g. osmotic) process. In figure 4*b*, cases $k_2 = 0$ and $k_2 \neq 0$ are compared for region 1. We observe that $k_2 \neq 0$ leads to the recovery of the usual diffusion in the asymptotic limit, whereas case $k_2 = 0$ results in a subdiffusive regime. Similar to the previous case, we verify that the particles present in region 1 manifest different regimes due to the membrane. In particular, for $k_2 \neq 0$, the asymptotic limit lead us to a regime governed by the bulk equation with $\langle(\Delta_1 x)^2\rangle \to t^{\alpha_1}$ for region 1. For region 2, the spreading of the system is asymptotically governed by $\langle(\Delta_2 x)^2\rangle \to t^{(\alpha_2+\alpha_1)/2}$. This result depends on α_2 and α_1, in contrast with the cases discussed for $k_2 = 0$ in figures 2 and 3 which only depends on α_2 in the asymptotic limit and the sorption—desorption processes manifested by the membrane.

3. Discussion and conclusion

We have investigated the behaviour of the particle distribution in the bulk and at the interface of a heterogeneous system separated by a membrane. The substance can be sorbed and desorbed by the surface. It can be also transported from one region to other. The results obtained have shown a rich class of behaviours and how the bulk and the membrane may influence the behaviour of the system. In particular, from figures 3*a* and 4*b*, we observe that this influence on diffusion may be responsible for the existence of different diffusive regimes for the particles in region 2 (as can be seen in figure 3*a*) or for different asymptotic regimes of the particle distribution in region 1 if a reverse transport process is allowed (as shown in figure 4*b*). The results presented here may find applications, for instance, to investigate the behaviour of living cells due to the many peculiarities of these systems, such as sources of obstruction and binding that happen in these high density and viscosity media occurring because of molecular crowding. In particular, diffusion in biological systems is observed to be orders of magnitude slower than the one predicted by the usual diffusion theory [44], which has led authors to incorporate anomalous diffusion when modelling diffusion on cell membranes (e.g. [45–47]). The model has been formulated here as general as possible to be applied to a wide variety of experimentally relevant situations.

Data accessibility. The figures were done by performing numerical calculations by using equations (2.17) (after performing the inverse Laplace transform), (2.20) and (2.21).

Authors' contributions. All the authors conducted the theoretical and numerical analysis and prepared the manuscript.

Competing interests. We declare we have no competing interests.

Funding. The authors thank CNPq (Brazilian agency) for the partial financial support, in particular, under grants nos 303642/2014-9 (E.K.L.), 462067/2014-9 (A.A.T.) and 440650/2014-3 (H.V.R.).

References

1. Ashrafuzzaman M, Tuszynski JA. 2013 *Membrane biophysics*. Heidelberg, Germany: Springer.
2. Hsieh HP. 1996 *Inorganic membranes for separation and reaction*. Amsterdam, The Netherlands: Elsevier.
3. Alberts B, Johnson A, Lewis J, Raff M, Roberts K, Walter P. 2002 *Molecular biology of the cell*. New York, NY: Garland Science.
4. Wall J. 2006 *Guyton and hall textbook of medical physiology*. Philadelphia, PA: Elsevier.
5. Edidin M. 2003 Lipids on the frontier: a century of cell-membrane bilayers. *Nature* **4**, 414–418. (doi:10.1038/nrm1102)
6. Hristov J. 2016 A unified nonlinear fractional equation of the diffusion-controlled surfactant adsorption: reappraisal and new solution of the Ward-Tordai problem. *J. King Saud Univ. Sci.* **28**, 7–13. (doi:10.1016/j.jksus.2015.03.008)
7. Hristov J. 2015 Approximate solutions to time-fractional models by integral balance approach. In *Fractional dynamics* (eds C Cattani, HM Srivastava, X-J Yang), pp. 78–109. Warsaw, Poland: De Gruyter Open.
8. Shannon MA, Bohn PW, Elimelech M, Georgiadis JG, Marinas BJ, Mayes AM. 2008 Science and technology for water purification in the coming decades. *Nature* **452**, 301–310. (doi:10.1038/nature06599)
9. Lewis SR, Datta S, Guia M, Coker EL, Hugginsa FE, Daunert S, Bachas L, Bhattacharyya D. 2011 Reactive nanostructured membranes for water purification. *Proc. Natl Acad. Sci. USA* **108**, 8577–8582. (doi:10.1073/pnas.1101144108)
10. Korabel N, Barkai E. 2010 Paradoxes of subdiffusive infiltration in disordered systems. *Phys. Rev. Lett.* **104**, 170603. (doi:10.1103/PhysRevLett.104.170603)
11. Korabel N, Barkai E. 2011 Anomalous infiltration. *J. Stat. Mech.* **2011**, P05022. (doi:10.1088/1742-5468/2011/05/P05022)
12. Kosztolowicz T. 2015 Random walk model of subdiffusion in a system with a thin membrane. *Phys. Rev. E* **91**, 022102. (doi:10.1103/PhysRevE.91.022102)

13. Guimaraes VG, Ribeiro HV, Li Q, Evangelista LR, Lenzi EK, Zola RS. 2015 Unusual diffusing regimes caused by different adsorbing surfaces. *Soft Matter* **11**, 1658–1666. (doi:10.1039/c5sm00151j)
14. Zaburdaev V, Denisov S, Klafter J. 2015 Lévy walks. *Rev. Mod. Phys.* **87**, 483. (doi:10.1103/RevModPhys.87.483)
15. Klages R, Radons G, Sokolov IM. 2008 *Anomalous transport: foundations and applications*. Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA.
16. Barkai E. 2001 Fractional Fokker-Planck equation, solution, and application. *Phys. Rev. E* **63**, 046118. (doi:10.1103/PhysRevE.63.046118)
17. Metzler R, Klafter J. 2004 The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A Math. Gen.* **37**, R161–R208. (doi:10.1088/0305-4470/37/31/R01)
18. Klafter J, Sokolov IM. 2011 *First steps in random walks: from tools to applications*. Oxford, UK: Oxford University Press.
19. Metzler R, Jeon J-H, Cherstvy AG, Barkai E. 2014 Anomalous diffusion models and their properties: non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Phys. Chem. Chem. Phys.* **16**, 24 128–24 164. (doi:10.1039/c4cp03465a)
20. Cherstvy AG, Chechkin AV, Metzler R. 2014 Particle invasion, survival, and non-ergodicity in 2D diffusion processes with space-dependent diffusivity. *Soft Matter* **10**, 1591–1601. (doi:10.1039/C3SM52846D)
21. Höfling F, Franosch T. 2013 Anomalous transport in the crowded world of biological cells. *Rep. Prog. Phys.* **76**, 046602. (doi:10.1088/0034-4885/76/4/046602)
22. Bressloff PC. 2014 *Stochastic processes in cell biology*. Heidelberg, Germany: Springer.
23. Snopok BA. 2014 Nonexponential kinetics of surface chemical reactions. *Theor. Exp. Chem.* **50**, 67–95. (doi:10.1007/s11237-014-9351-0)
24. dePaula JL, Santoro PA, Zola RS, Lenzi EK, Evangelista LR, Ciuchi F, Mazzulla A, Scaramuzza N. 2012 Non-Debye relaxation in the dielectric response of nematic liquid crystals: surface and memory effects in the adsorption-desorption process of ionic impurities. *Phys. Rev. E* **86**, 051705. (doi:10.1103/PhysRevE.86.051705)
25. Meztler R, Klafter J. 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**, 1–77. (doi:10.1016/S0370-1573(00)00070-3)
26. Podlubny I. 1999 *Fractional differential equations*. San Diego, CA: Academic Press.
27. Mathai AM, Saxena RK, Haubold HJ. 2009 *The H-function: theory and applications*. New York, NY: Springer.
28. Alaimo G, Zingales M. 2015 Laminar flow through fractal porous materials: the fractional-order transport. *Commun. Nonlinear Sci.* **22**, 889–902. (doi:10.1016/j.cnsns.2014.10.005)
29. Santoro PA, de Paula JL, Lenzi EK, Evangelista LR. 2011 Anomalous diffusion governed by a fractional diffusion equation and the electrical response of an electrolytic cell. *J. Chem. Phys.* **135**, 114704. (doi:10.1063/1.3637944)
30. Hanyga A. 2002 Multi-dimensional solutions of space-time-fractional diffusion equations. *Proc. R. Soc. Lond. A* **458**, 429–450. (doi:10.1098/rspa.2001.0893)
31. Zola RS, Lenzi EK, Evangelista LR, Barbero G. 2007 Memory effect in the adsorption phenomena of neutral particles. *Phys. Rev. E* **75**, 042601. (doi:10.1103/PhysRevE.75.042601)
32. Kopelman R. 1998 Fractal reaction kinetics. *Science* **241**, 1620–1626. (doi:10.1126/science.241.4873.1620)
33. Schnell S, Turner TE. 2004 Reaction kinetics in intracellular environments with macro-molecular crowding: simulations and rate laws. *Prog. Biophys. Mol. Biol.* **85**, 235–260. (doi:10.1016/j.pbiomolbio.2004.01.012)
34. Berry H. 2002 Monte carlo simulations of enzyme reactions in two dimensions: fractal kinetics and spatial segregation. *Biophys. J.* **83**, 1891–1901. (doi:10.1016/S0006-3495(02)73953-2)
35. Grima R, Schnell S. 2006 A systematic investigation of the rate laws valid in intracellular environments. *Biophys. Chem.* **124**, 1–10. (doi:10.1016/0301-0104(92)87149-4)
36. Klann MT, Lapin A, Reuss M. 2011 Agent-based simulation of reactions in the crowded and structured intracellular environment: influence of mobility and location of the reactants. *BMC Syst. Biol.* **5**, 71. (doi:10.1186/1752-0509-5-71)
37. Berberan-Santos MN, Martinho JMG. 1992 A linear response approach to kinetics with time-dependent rate coefficients. *Chem. Phys.* **164**, 259–269. (doi:10.1016/0301-0104(92)87149-4)
38. Voit EO, Martens HA, Omholt SW. 2015 150 years of the mass action law. *PLoS Comput. Biol.* **11**, e1004012. (doi:10.1371/journal.pcbi.1004012)

39. Grima R, Schnell S. 2006 How reaction kinetics with time-dependent rate coefficients differs from generalized mass action. *Chem. Phys. Chem.* **7**, 1422–1424. (doi:10.1002/cphc.200600081)
40. Parsegian VA. 2006 *Van der Waals forces: a handbook for biologists, chemists, engineers, and physicists*. Cambridge, UK: Cambridge University Press.
41. Sokolov IM, Schmidt MGW, Sagués F. 2006 Reaction-subdiffusion equations. *Phys. Rev. E* **73**, 031102. (doi:10.1103/PhysRevE.73.031102)
42. Lomholt MA, Zaid IM, Metzler R. 2007 Subdiffusion and weak ergodicity breaking in the presence of a reactive boundary. *Phys. Rev. Lett.* **98**, 200603. (doi:10.1103/PhysRevLett.98.200603)
43. Zaid IM, Lomholt MA, Metzler R. 2009 How subdiffusion changes the kinetics of binding to a surface. *Biophys. J.* **97**, 710–721. (doi:10.1016/j.bpj.2009.05.022)
44. Fujiwara T, Ritchie K, Murakoshi H, Jacobson K, Kusumi A. 2002 Phospholipids undergo hop diffusion in compartmentalized cell membrane. *J. Cell. Biol.* **157**, 1071–1081. (doi:10.1083/jcb.200202050)
45. Saxton MJ. 1996 Anomalous diffusion due to binding: a Monte Carlo study. *Biophys. J.* **70**, 1250–1262. (doi:10.1016/S0006-3495(96)79682-0)
46. Banks DS, Fradin C. 2005 Anomalous diffusion of proteins due to molecular crowding. *Biophys. J.* **89**, 2960–2971. (doi:10.1529/biophysj.104.051078)
47. Nicolau Jr DV, Hancock JF, Burrage K. 2007 Sources of anomalous diffusion on cell membranes: a Monte Carlo study. *Biophys. J.* **92**, 1975–1987. (doi:10.1529/biophysj.105.076869)