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# Fractional diffusion equations coupled by reaction terms

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## HIGHLIGHTS

- Solutions for a set of fractional diffusion equations with reaction terms.
- Interplay between different diffusive regimes and anomalous diffusion.
- Asymptotic behavior governed by long tailed distributions.

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## ABSTRACT

We investigate the behavior for a set of fractional reaction–diffusion equations that extend the usual ones by the presence of spatial fractional derivatives of distributed order in the diffusive term. These equations are coupled via the reaction terms which may represent reversible or irreversible processes. For these equations, we find exact solutions and show that the spreading of the distributions is asymptotically governed by the same the long-tailed distribution. Furthermore, we observe that the coupling introduced by reaction terms creates an interplay between different diffusive regimes leading us to a rich class of behaviors related to anomalous diffusion.



## 1. Introduction

Anomalous diffusion has been reported in several contexts such as fractal globules [1], molecular crowding [2], particles transported by parallel flows [3], brain metabolites [4], and particle tracking [5]. One of the main characteristics of this phenomenon is the unusual time dependence of the mean square displacement that can be, *e.g.*, nonlinear $\langle (x - \langle x \rangle)^2 \rangle \sim t^{\alpha}$, with $\alpha < 1$ and $\alpha > 1$ corresponding to sub- and superdiffusion, or not well defined as occurs with Lévy distributions, which are characterized by a long-tailed distributions. In this context, the fractional calculus has proved to be a powerful tool in scenarios such as tumor growth [6], electrical response [7], solute transport [8], living cells [9], anomalous transport in biological cells [10], solar cosmic-ray transport [11], and diffusion-controlled surfactant adsorption [12] by extending the usual approaches to noninteger differential operators. Typical examples are the fractional diffusion equations [13], which display a wide variety of behaviors that can be connected to anomalous diffusion. The fractional equations are also related to random walks, where the fractional operators appear as a consequence of the behavior exhibited by the waiting time and/or the jumping probabilities distributions, generally characterized by long-tailed distributions. These distributions cause the

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divergence of low-order moments, and consequently the hypotheses underlying the central limit theorem are not valid. In particular, fractional order in the spatial derivative (Laplacian) of a diffusion equation can be connected to Lévy distributions in the jumping probabilities, leading to a divergent behavior in the variance. Another interesting point is the behavior obtained from these equations when reaction terms are considered [14–23]. Reaction terms have been used to investigate several situations related to pattern formation [24,25], fluorescence recovery after photobleaching [26], non-Fickian phenomena related to astronomical phenomena [27], solute transport subject to bimolecular reactions [28], and ecology [29]. Thus, the analysis of fractional diffusion equations, especially those with reaction terms, is of the great interest from the analytical and numerical point of view due to the large number of scenarios which can be connected.

Here, we investigate a two component system governed by fractional diffusion equations with reaction terms, which may represent a reversible or irreversible process depending on the choice of the rates. In particular, we consider the following equations

$$\frac{\partial}{\partial t}\rho_1(x,t)=\int_1^{\mu_1}d\overline{\mu}_1\mathcal{D}_1(\overline{\mu}_1)\frac{\partial^{\overline{\mu}_1}}{\partial|x|^{\overline{\mu}_1}}\rho_1(x,t)-k_1\rho_1(x,t)+k_2\rho_2(x,t), \tag{1}$$

$$\frac{\partial}{\partial t}\rho_2(x,t)=\int_1^{\mu_2}d\overline{\mu}_2\mathcal{D}_2(\overline{\mu}_2)\frac{\partial^{\overline{\mu}_2}}{\partial|x|^{\overline{\mu}_2}}\rho_2(x,t)+k_1\rho_1(x,t)-k_2\rho_2(x,t), \tag{2}$$

where $1<\mu_1\le 2$ and $1<\mu_2\le 2$. The constants $k_1$ and $k_2$ may be related to reversible (forward and backward) reactions, *i.e.*, $1\rightleftharpoons 2$, or irreversible reactions, *i.e.*, $1\to 2$. The fractional operator is the Riesz–Weyl [30] one, which encompasses situations characterized by long-tailed distributions in connection to the Lévy distributions. The functions $\mathcal{D}_1(\overline{\mu}_1)$ and $\mathcal{D}_2(\overline{\mu}_2)$ are the distributions related to the index $\overline{\mu}_1$ and $\overline{\mu}_2$. In particular, we shall consider that $\mathcal{D}_1(\overline{\mu}_1)=\mathcal{D}_{\mu_1'}\delta(\overline{\mu}_1-\mu_1')+\mathcal{D}_{\mu_1}\delta(\overline{\mu}_1-\mu_1)$ and $\mathcal{D}_2(\overline{\mu}_2)=\mathcal{D}_{\mu_2'}\delta(\overline{\mu}_2-\mu_2')+\mathcal{D}_{\mu_2}\delta(\overline{\mu}_2-\mu_2)$. This choice leads us to an interplay between the regimes characterized by the different index, $\mu_{1(2)}$ and $\mu_{1(2)}'$. In this scenario, a reaction process occurs, and consequently the regime manifested by one processes influences the other due to the coupling produced by the reaction terms. In Section 2, we find exact solutions for this set of equations and discuss the spreading of the distributions in different scenarios. The discussions and conclusions are presented in Section 3.

## 2. Diffusion and reaction

We investigate the behavior of Eqs. (1) and (2) by considering different scenarios. We first consider the case $\mathcal{D}_1(\overline{\mu}_1)\neq 0$ and $\mathcal{D}_2(\overline{\mu}_2)=0$, followed by the case in which $\mathcal{D}_1(\overline{\mu}_1)\neq 0$ and $\mathcal{D}_2(\overline{\mu}_2)\neq 0$. In both cases, we obtain exact solutions in terms of the Fox H-functions [30] and analyze the spreading of the solutions.

### 2.1. The case $\mathcal{D}_1(\overline{\mu}_1)\neq 0$ and $\mathcal{D}_2(\overline{\mu}_2)=0$

We start the case $\mathcal{D}_1(\overline{\mu}_1)\neq 0$ and $\mathcal{D}_2(\overline{\mu}_2)=0$ by assuming an infinite medium where all relevant quantities are diffusing in one dimension, *i.e.*, the $x$ direction with $-\infty<x<\infty$. Eqs. (1) and (2) for this case can be written as

$$\frac{\partial}{\partial t}\rho_1(x,t)=\int_1^{\mu_1}d\overline{\mu}_1\mathcal{D}_1(\overline{\mu}_1)\frac{\partial^{\overline{\mu}_1}}{\partial|x|^{\overline{\mu}_1}}\rho_1(x,t)-k_1\rho_1(x,t)+k_2\rho_2(x,t), \tag{3}$$

$$\frac{\partial}{\partial t}\rho_2(x,t)=k_1\rho_1(x,t)-k_2\rho_2(x,t). \tag{4}$$

These equations represent a process where one substance may be sorbed or chemically react with another kind of substance that is diffusing. Thus, this case corresponds to a diffusion problem in which some of the diffusing substance is immobilized as the diffusion proceeds, or as a problem in chemical kinetic in which the rate of the reaction depends on the supply rate of one of the reactants by diffusion. In both situations, the quantities $k_1$ and $k_2$ are relevant to define the rate related to these processes. For these equations, it is possible to show that

$$\int_{-\infty}^{\infty}dx\rho_1(x,t)+\int_{-\infty}^{\infty}dx\rho_2(x,t)=\text{const.}, \tag{5}$$

implying that the number of particles is conserved in this system. In order to find the solution for this set of equations, from the formal point of view, we may use the Fourier and Laplace transform yielding

$$\rho_1(k,s)=\frac{(s+k_2)\,\rho_1(k,0)}{s(s+k_1+k_2)+(s+k_2)\mathcal{I}_1(k;\mu_1,\mu_1')}, \tag{6}$$

$$\rho_2(k,s)=\frac{k_1}{s+k_2}\rho_1(k,s) \tag{7}$$

for $\mathcal{I}_1(k;\mu_1,\mu_1') = \int_1^{\mu_1} d\overline{\mu}_1 \mathcal{D}_1(\overline{\mu}_1)|k|^{\overline{\mu}_1}$ and an arbitrary initial condition $\rho_1(x,0)$ with $\int_{-\infty}^{\infty}\rho_1(x,0)dx = 1$ and $\rho_2(x,0) = 0$. Performing the inverse Fourier and Laplace transforms for $\mathcal{D}_1(\overline{\mu}_1) = \mathcal{D}_{\mu_1'}\delta(\overline{\mu}_1-\mu_1') + \mathcal{D}_{\mu_1}\delta(\overline{\mu}_1-\mu_1)$, we obtain that the solution for $\rho_1(x,t)$ is given by

$$\rho_1(x,t) = \int_{-\infty}^{\infty}\rho_1(x',0)\mathcal{G}_{\mathcal{I}_1}(x-x',t)dx' + \int_{-\infty}^{\infty}\rho_1(x',0)\Phi_1(x-x',t)dx', \tag{8}$$

where

$$\mathcal{G}_{\mathcal{I}_1}(x,t) = \int_{-\infty}^{\infty}\frac{dx'}{\mu_1\mu_1'\left(\mathcal{D}_{\mu_1}t\right)^{1/\mu_1}\left(\mathcal{D}_{\mu_1'}t\right)^{1/\mu_1'}} \mathrm{H}_{2,2}^{1,1}\left[\frac{|x'|}{\left(\mathcal{D}_{\mu_1}t\right)^{1/\mu}}\left|\begin{matrix}\left(1-\frac{\mu_1}{2},\frac{\mu_1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right] \times \mathrm{H}_{2,2}^{1,1}\left[\frac{|x-x'|}{\left(\mathcal{D}_{\mu_1'}t\right)^{1/\mu_1'}}\left|\begin{matrix}\left(1-\frac{\mu_1'}{2},\frac{\mu_1'}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right] \tag{9}$$

is the Green function for the free case for $\rho_1(x,t)$ when the reaction terms are absent and

$$\Phi_1(x,t) = \sum_{n=1}^{\infty}(-1)^n\int_{-\infty}^{\infty}dx_n\int_0^t dt_n\Delta_1(x-x_n,t-t_n)\cdots\int_{-\infty}^{\infty}dx_1\int_0^t dt_1\Delta_1(x_2-x_1,t_2-t_1)\mathcal{G}_{\mathcal{I}_1}(x_1,t_1) \tag{10}$$

$$\Delta_1(x,t) = k_1\mathcal{G}_{\mathcal{I}_1}(x,t) - k_1k_2\int_0^t dt e^{-k_2t'}\mathcal{G}_{\mathcal{I}_1}(x,t-t') \tag{11}$$

is the contribution of the reaction terms present in Eqs. (6) and (7). Note the Green function is expressed in terms of the Fox H-function [30], $\mathrm{H}_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p,A_p)\\(b_q\ B_q)\end{matrix}\right.\right]$, a function that is usually connected to anomalous relaxation of the system due to the presence of spatial fractional derivative. In addition, the choice of the function $\mathcal{D}_1(\overline{\mu}_1)$ takes into account the interplay between two different diffusive regimes characterized by $\mu_1$ and $\mu_1'$. Others choices for $\mathcal{D}_1(\overline{\mu}_1)$ are also possible and lead to different scenarios. The solution for the component (substance) obtained from the reaction process is given by

$$\rho_2(x,t) = k_1e^{-k_2t}\int_0^t e^{k_2t'}\rho_1(x,t')dt'. \tag{12}$$

By using the previous results, it is possible to obtain the quantity of particles present in the bulk for each component, which is given by $\mathcal{S}_1(t) = \int_{-\infty}^{\infty}dx\rho_1(x,t) = (k_2+k_1e^{-(k_1+k_2)t})/(k_2+k_1)$ and $\mathcal{S}_2(t) = \int_{-\infty}^{\infty}dx\rho_2(x,t) = k_1(1-e^{-(k_1+k_2)t})/(k_2+k_1)$ where $\mathcal{S}_1(t)+\mathcal{S}_2(t) = 1$.

Fig. 1 illustrates the behavior of the distribution for different values of $k_1$ and $k_2$. Fig. 2 shows the behavior of $1/\rho_1^2(0,t)$ and $1/\rho_2^2(0,t)$ as a measure of the spreading of these distributions. For $1/\rho_1^2(0,t)$, we observe the interplay between the usual ($\mu_1 = 2$) and anomalous behavior ($\mu_1' = 3/2$). The effect of the reaction process is also noticeable in this figure. In particular, we note the influence of the reaction on the diffusive regimes governed by the index $\mu_1$ and $\mu_1'$ depending on the value of $k_1$ and $k_2$. For $k_1, k_2 \gg 1$, the effects of these constants on the spreading of the system are expected to be manifested for small times. On the other hand, the effect of $k_1, k_2 \ll 1$ is manifested for long times. For $1/\rho_2^2(0,t)$, we observe an interesting behavior, where the function initially decreases and after increases. The first behavior is related to the formation of this substance by the reaction process, implying that $\rho_2(0,t)$ first increases (see Fig. 3). After, $\rho_2(x,t)$ starts also to produce the other, $\rho_1(x,t)$, by the reaction process, and consequently spreads as the time evolves, leading us to the second regime manifested by $1/\rho_2^2(0,t)$. It is worth noting that the diffusive regime manifested by $1/\rho_2^2(0,t)$ has asymptotically the same dependence of the $1/\rho_1^2(0,t)$. In fact, for the case of Fig. 1, one can show that $\rho_1(0,t)\sim k_2/((k_1+k_2)^{1-1/\mu_1'}(k_2\mathcal{D}_{\mu_1'}t)^{1/\mu_1'})$ and $\rho_2(0,t)\sim k_1/((k_1+k_2)^{1-1/\mu_1'}(k_2\mathcal{D}_{\mu_1'}t)^{1/\mu_1'})$ in the asymptotic limit of long times.

### *2.2. The case: $\mathcal{D}_1(\overline{\mu}_1)\neq 0$ and $\mathcal{D}_2(\overline{\mu}_2)\neq 0$*

We now discuss the case in which both components are diffusing and can react with rates determined by $k_1$ and $k_2$. Following the procedure employed in the previous section, we start by using the Fourier transform to overcome some complex calculations. After applying this integral transform, we obtain that Eqs. (1) and (2) can be written as

$$\frac{\partial}{\partial t}\rho_1(k,t) = -\mathcal{I}_1(k;\mu_1,\mu_1')\rho_1(k,t) - k_1\rho_1(k,t) + k_2\rho_2(k,t), \tag{13}$$

$$\frac{\partial}{\partial t}\rho_2(k,t) = -\mathcal{I}_2(k;\mu_2,\mu_2')\rho_2(x,t) + k_1\rho_1(x,t) - k_2\rho_2(x,t), \tag{14}$$

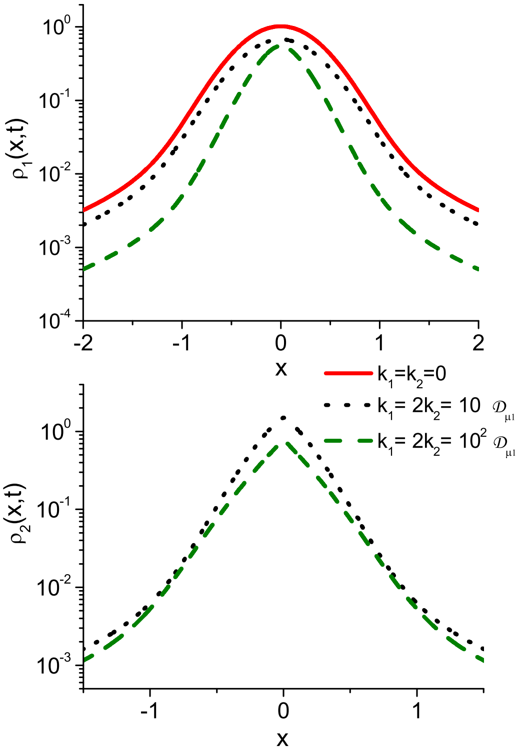


**Fig. 1.** Behavior of Eqs. (8) and (12) for different values of $k_1$ and $k_2$ with $\mu_1 = 2$ and $\mu_1' = 3/2$. We consider, for simplicity, $\mathcal{D}_{\mu_1} = \mathcal{D}_{\mu_1'}$, $\rho_1(x, 0) = \delta(x)$, $\rho_2(x, 0) = 0$, and $t = 5 \times 10^{-2}/\mathcal{D}_{\mu_1}$.

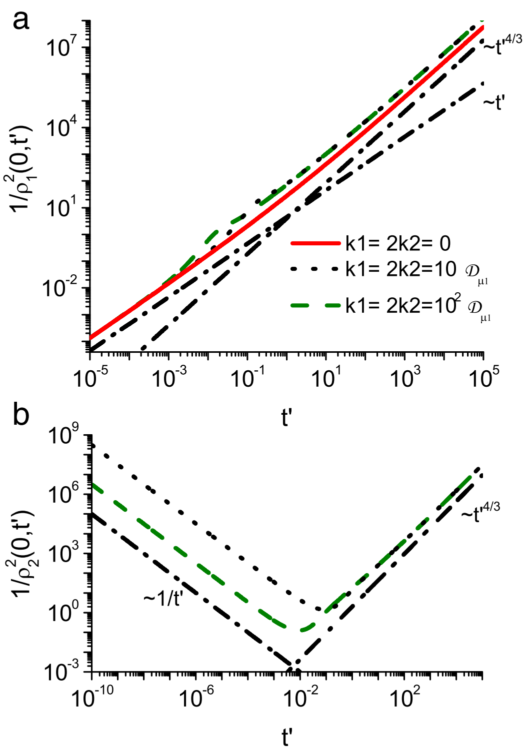


**Fig. 2.** Behavior of $1/\rho_1^2(0, t')$ and $1/\rho_2^2(0, t')$ versus $t' = \mathcal{D}_{\mu_1} t$ for different values of $k_1$ and $k_2$ with $\mu_1 = 2$ and $\mu_1' = 3/2$. We consider, for simplicity, $\mathcal{D}_{\mu_1} = \mathcal{D}_{\mu_1'}$, $\rho_1(x, 0) = \delta(x)$, and $\rho_2(x, 0) = 0$.

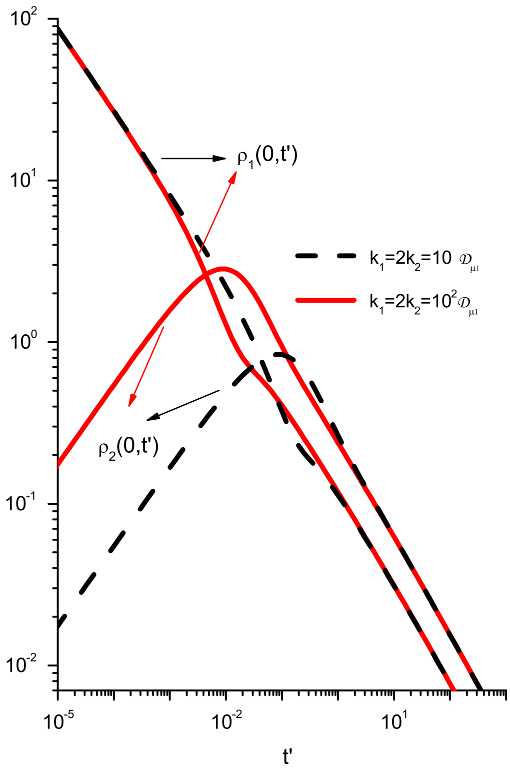


**Fig. 3.** Behavior of $\rho_1(0,t')$ and $\rho_2(0,t')$ versus $t' = \mathcal{D}_2 t$ for different values of $k_1$ and $k_2$ with $\mu_1 = 2$ and $\mu_1' = 3/2$. We consider, for simplicity, $\mathcal{D}_{\mu_1} = \mathcal{D}_{\mu_1'}$, $\rho_1(x,0) = \delta(x)$, and $\rho_2(x,0) = 0$.

with $\mathcal{I}_1(k;\mu_1,\mu_1') = \int_1^{\mu_1} d\overline{\mu}_1 \mathcal{D}_1(\overline{\mu}_1)|k|^{\overline{\mu}_1}$ and $\mathcal{I}_2(k;\mu_2,\mu_2') = \int_1^{\mu_2} d\overline{\mu}_2 \mathcal{D}_2(\overline{\mu}_2)|k|^{\overline{\mu}_2}$. Notice that Eq. (5) also holds for this case, *i.e.*, the total number of particles (mass) is a constant. For this set of equations, we consider as initial condition $\rho_1(x,0)$ and $\rho_2(x,0)$ arbitrary with $\int_{-\infty}^{\infty}(\rho_1(x,0)+\rho_2(x,0))\,dx = 1$. After performing the Laplace transform and some calculations, we obtain that

$$\begin{aligned}\rho_1(k,s) &= \frac{\left(s+\mathcal{I}_2(k;\mu_2,\mu_2')+k_2\right)\rho_1(k,0)}{(s+k_2+\mathcal{I}_2(k;\mu_2,\mu_2'))(s+\mathcal{I}_1(k;\mu_1,\mu_1'))+k_1(s+\mathcal{I}_2(k;\mu_2,\mu_2'))},\\ &+\frac{k_2\rho_2(k,0)}{(s+k_2+\mathcal{I}_2(k;\mu_2,\mu_2'))(s+\mathcal{I}_1(k;\mu_1,\mu_1'))+k_1(s+\mathcal{I}_2(k;\mu_2,\mu_2'))} \end{aligned} \tag{15}$$

$$\rho_2(k,s) = \frac{k_1\rho_1(k,s)}{s+\mathcal{I}_2(k;\mu_2,\mu_2')+k_2} + \frac{\rho_2(k,0)}{s+\mathcal{I}_2(k;\mu_2,\mu_2')+k_2}. \tag{16}$$

Applying the inverse Fourier and Laplace transform in the previous equations by taking into account $\mathcal{D}(\overline{\mu}_1) = \mathcal{D}_{\mu_1'}\delta(\overline{\mu}_1 - \mu_1') + \mathcal{D}_{\mu_1}\delta(\overline{\mu}_1 - \mu_1)$ and $\mathcal{D}(\overline{\mu}_2) = \mathcal{D}_{\mu_2'}\delta(\overline{\mu}_2 - \mu_2') + \mathcal{D}_{\mu_2}\delta(\overline{\mu}_2 - \mu_2)$, we obtain that the solution for $\rho_1(x,t)$ is given by

$$\begin{aligned}\rho_1(x,t) &= \int_{-\infty}^{\infty}\rho_1(x,0)\left(\mathcal{G}_{\mathcal{I}_1}(x-x',t)+\Upsilon_{1,2}(x-x',t)\right)dx'\\ &- k_2\int_0^t\int_{-\infty}^{\infty}\mathcal{P}(x',t')\left(\mathcal{G}_{\mathcal{I}_1}(x-x',t-t')+\Upsilon_{1,2}(x-x',t-t')\right)dx'dt',\end{aligned} \tag{17}$$

where $\mathcal{P}(x,t) = \int_{-\infty}^{\infty} dx'\rho_2(x',0)\mathcal{G}_{\mathcal{I}_2}(x-x',t)$,

$$\Upsilon_{1,2}(x,t) = \sum_{n=1}^{\infty}(-1)^n\int_{-\infty}^{\infty}dx_n\int_0^t dt_n\,\Xi_{1,2}(x-x_n,t-t_n)\cdots\int_{-\infty}^{\infty}dx_1\int_0^t dt_1\,\Xi_{1,2}(x_2-x_1,t_2-t_1)\mathcal{G}_{\mathcal{I}_1}(x_1,t_1) \tag{18}$$

$$\Xi_{1,2}(x,t) = k_1\mathcal{G}_{\mathcal{I}_1}(x,t) - k_1k_2\int_0^t dt e^{-k_2t'}\int_{-\infty}^{\infty}dx'\mathcal{G}_{\mathcal{I}_2}(x-x',t')\mathcal{G}_{\mathcal{I}_1}(x,t-t') \tag{19}$$

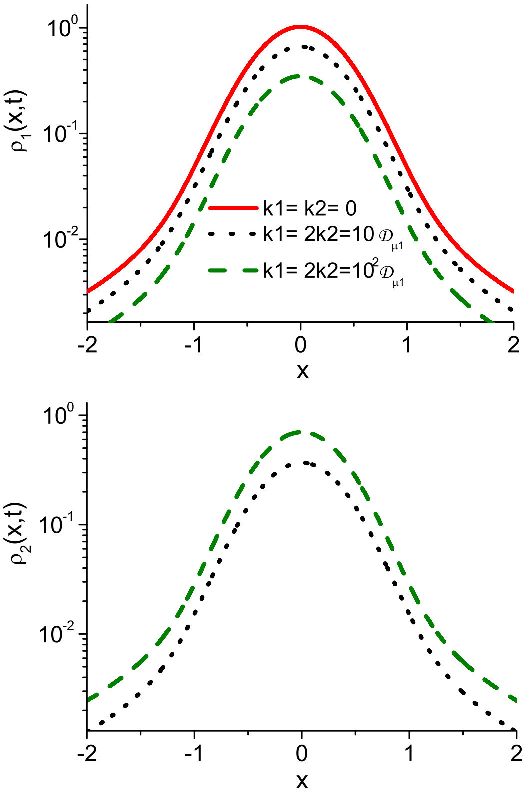


**Fig. 4.** Behavior of Eqs. (17) and (22) for different values of $k_1$ and $k_2$ with $\mu_1 = \mu_2 = 2$, $\mu'_1 = 3/2$, and $\mu'_2 = 5/4$. We also consider, for simplicity, $\mathcal{D}_{\mu_1} = \mathcal{D}_{\mu'_1} = \mathcal{D}_{\mu_2} = \mathcal{D}_{\mu'_2}$, $\rho_1(x, 0) = \delta(x)$, $\rho_2(x, 0) = 0$, and $t = 5 \times 10^{-2}\mathcal{D}_{\mu_1}$.

and

$$\mathcal{G}_{\mathcal{I}_2}(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}x'}{\mu_2\mu'_2\left(\mathcal{D}_{\mu_2}t\right)^{1/\mu_2}\left(\mathcal{D}_{\mu'_2}t\right)^{1/\mu'_2}} \mathrm{H}^{1,1}_{2,2}\left[\frac{|x'|}{\left(\mathcal{D}_{\mu_2}t\right)^{1/\mu_2}}\left|\begin{matrix}\left(1-\frac{\mu_2}{2},\frac{\mu_2}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right] \tag{20}$$

$$\times \mathrm{H}^{1,1}_{2,2}\left[\frac{|x-x'|}{\left(\mathcal{D}_{\mu'_2}t\right)^{1/\mu'_2}}\left|\begin{matrix}\left(1-\frac{\mu'_2}{2},\frac{\mu'_2}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1),\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right] \tag{21}$$

is the Green function of the free case for $\rho_2(x,t)$ in absence of reaction terms. For $\rho_2(x,t)$, we obtain that

$$\rho_2(x,t) = \int_{-\infty}^{\infty}\rho_2(x',0)\mathcal{G}_{\mathcal{I}_2}(x-x',t)\mathrm{d}x' + k_1\int_{-\infty}^{\infty}\int_0^t \rho_1(x',t')\mathcal{G}_{\mathcal{I}_2}(x-x',t-t')\mathrm{d}t'\mathrm{d}x'. \tag{22}$$

The previous expressions for $\mathcal{D}_{\mu_1}(\overline{\mu}_1) = \mathcal{D}_{\mu_2}(\overline{\mu}_2) = \mathcal{D}_{\mu}(\overline{\mu})$ may be simplified to

$$\begin{aligned}\rho_1(x,t) &= \int_{-\infty}^{\infty}\mathrm{d}x'\rho_1(x',0)\mathrm{e}^{-(k_1+k_2)t}\mathcal{G}_{\mathcal{I}}(x-x',t)\mathrm{d}x' \\ &+ \frac{k_2}{k_1+k_2}\int_{-\infty}^{\infty}\left(\rho_1(x',0)+\rho_2(x',0)\right)(1-\mathrm{e}^{-(k_1+k_2)t})\mathcal{G}_{\mathcal{I}}(x-x',t)\mathrm{d}x',\end{aligned} \tag{23}$$

$$\begin{aligned}\rho_2(x,t) &= \int_{-\infty}^{\infty}\mathrm{d}x'\rho_2(x',0)\mathrm{e}^{-(k_1+k_2)t}\mathcal{G}_{\mathcal{I}}(x-x',t)\mathrm{d}x' \\ &+ \frac{k_1}{k_1+k_2}\int_{-\infty}^{\infty}\left(\rho_1(x',0)+\rho_2(x',0)\right)(1-\mathrm{e}^{-(k_1+k_2)t})\mathcal{G}_{\mathcal{I}}(x-x',t)\mathrm{d}x',\end{aligned} \tag{24}$$

where $\mathcal{G}_{\mathcal{I}_1}(x,t) = \mathcal{G}_{\mathcal{I}_2}(x,t) = \mathcal{G}_{\mathcal{I}}(x,t)$.

Fig. 4 illustrates the behavior of the distributions $\rho_1(x,t)$ and $\rho_2(x,t)$ for different values of $k_1$ and $k_2$. In this figure, we also consider $\mu_1 = \mu_2 = 2$ and $\mu'_1 = 3/2$ and $\mu'_2 = 5/4$. For each diffusion equation, we have an interplay between

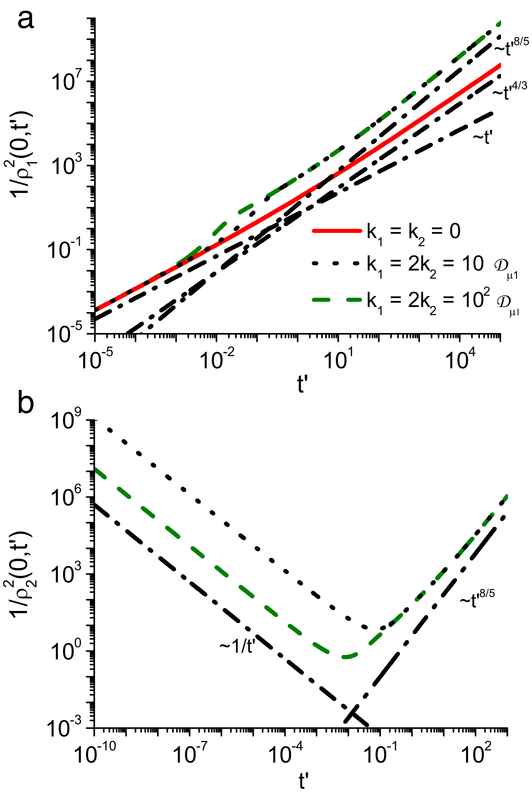


**Fig. 5.** Behavior of $1/\rho_1^2(0, t')$ and $1/\rho_2^2(0, t')$ versus $t' = \mathcal{D}_2 t$ for different values of $k_1$ and $k_2$ with $\mu_1 = \mu_2 = 2$, $\mu_1' = 3/2$, $\mu_2' = 5/4$. We also consider, for simplicity, $\mathcal{D}_2 = \mathcal{D}_{3/2} = \mathcal{D}_{5/4}$, $\rho_1(x, 0) = \delta(x)$, and $\rho_2(x, 0) = 0$.

the usual and anomalous diffusion. Fig. 5 shows that the time evolution of Eqs. (1) and (2) may be mutually modified by the different values of $\mu_1'$ and $\mu_2'$ due to the coupling produced by reaction terms. In particular, the asymptotic behavior is governed by the long-tailed distribution which is asymptotically characterized by $\mu_2' = 5/4$ in this case. Similar to case of the previous section, for the case of Fig. 4, one can show that $\rho_1(0, t) \sim k_2/((k_1 + k_2)^{1-1/\mu_2'}(k_1 \mathcal{D}_{\mu_2'} t)^{1/\mu_2'})$ and $\rho_2(0, t) \sim k_1/((k_1 + k_2)^{1-1/\mu_2'}(k_1 \mathcal{D}_{\mu_2'} t)^{1/\mu_2'})$ in the asymptotic limit of long times.

## 3. Discussion and conclusions

We investigated spatial fractional diffusion equations of distributed order with linear reactions terms. The reaction terms introduce a coupling between these equations and enable us to analyze situations characterized by a reversible ($k_1 \neq 0$ and $k_2 \neq 0$) or irreversible process ($k_1 = 0$ or $k_2 = 0$). We first analyzed the case characterized by one of them diffusing ($\mathcal{D}_1(\overline{\mu}_1) \neq 0$) and the other characterized by ($\mathcal{D}_2(\overline{\mu}_2) = 0$). For this case, we observed (Figs. 1 and 2) that the spreading of both distributions is governed by the long-tailed distribution obtained from $\mu_1' = 3/2$. The initial behavior (small times, $t \ll (\mathcal{D}_{\mu_1}/\mathcal{D}_{\mu_1'})^{\mu_1\mu_1'/(\mu_1-\mu_1')}$) for $\rho_1(x, t)$ is governed by a usual spreading, *i.e.*, $\rho_1(0, t) \sim 1/\sqrt{t}$, where $\mu_1 = 2$. For $\rho_2(x, t)$, we verified that $\rho_2(0, t)$ initially increases before decreasing. After this analysis, we considered the case $\mathcal{D}_1(\overline{\mu}_1) \neq 0$ with $\mathcal{D}_2(\overline{\mu}_2) \neq 0$, *i.e.*, a situation in which both components can diffuse. For this case, we also obtained exact solutions and showed that, due to the reaction terms coupling these equations, the asymptotic behavior is governed by a long-tailed distribution obtained by the smaller value of the set $\{\mu_1, \mu_1', \mu_2, \mu_2'\}$. This feature was evidenced in Fig. 5 when the index $\mu_1 = \mu_2 = 2$, $\mu_1' = 3/2$, and $\mu_2' = 5/4$ were considered. Finally, we hope that the results presented here will be useful in the diffusion of fractional diffusion equations and reaction processes.

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