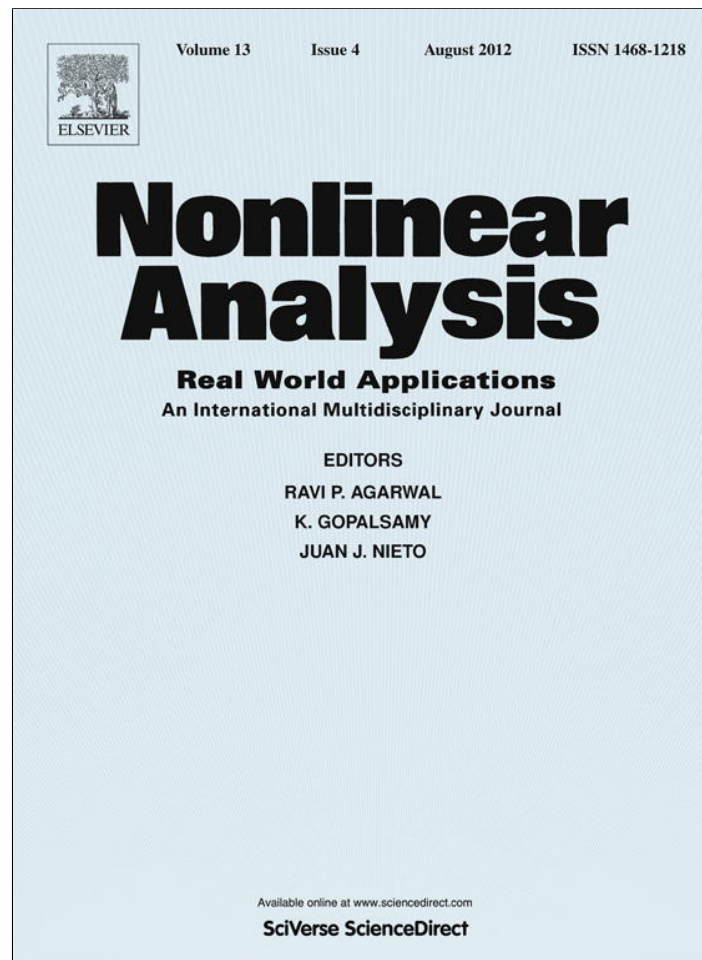


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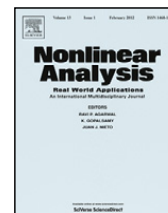
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Solutions for a fractional diffusion equation with noninteger dimensions

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ABSTRACT

We investigate a fractional diffusion equation with a nonlocal reaction term by using the Green function approach. We also consider a modified spatial operator in order to cover situations characterized by a noninteger dimension. The results show a unusual spreading of the initial condition which can be connected to a rich class of anomalous diffusive processes.

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1. Introduction

The non-Markovian characteristics of the processes related to anomalous diffusion [1–3] and its ubiquity in nature have motivated researchers of many scientific fields to investigate this subject. In fact, unusual processes are present in several situations such as diffusion on fractals [4,5], biological cells [6,7], animal foraging behavior [8], *human epidemic dynamic model* [9], nanoscience [10,11], geophysical systems [12,13], motion of colloidal particles [14], systems with long range interactions [15,16], and the adsorption–desorption process [17–19]. One of the main characteristics of the anomalous diffusion is the time dependence of the mean square displacement which, depending on the stochastic process, may be expressed, for example, as $\langle (r - \langle r \rangle)^2 \rangle \sim t^\alpha$, where $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ correspond to sub-, normal and superdiffusive cases, respectively. It is also possible to have situations in which the mean square displacement is not defined [1]. This feature occurs when Lévy distributions are involved. *In order to cover these scenarios, some formalisms such as the random walks [20], master equations [21], Langevin equations [2,22] and diffusion equations [23,24] have been used with suitable considerations. In particular, the diffusion equation has been extended by incorporating spatial and time fractional derivatives. A direct consequence is a unusual spreading of the system which may be related to rich classes of anomalous diffusion. They have been systematically investigated such as the developments performed in Refs. [25–36]. In this sense, the Schrödinger equation has also been extended as shown in Refs. [37–43] by incorporating fractional derivatives, α -dimensional fractional space [44,45], and nonlocal terms [46,47]. Furthermore, a Lippmann–Schwinger equation for the fractional quantum mechanics has been proposed in Ref. [48]. In the framework of these formal developments, here the solutions of the following diffusion equation will be considered:*

$$\frac{\partial^\gamma}{\partial t^\gamma} \rho(\mathbf{r}, t) = \mathcal{D} \tilde{\nabla}^2 \rho(\mathbf{r}, t) - \int_0^t d\bar{t} \alpha(t - \bar{t}) \rho(\mathbf{r}, \bar{t}), \quad (1)$$

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where $\mathbf{r} = (r, \theta)$, \mathcal{D} has the dimension $[L]^2/[T]^\gamma$ and for $\gamma = 1$, it recovers the usual dimension. The fractional time derivative considered in Eq. (1) is the Caputo one [49] and the spatial operator is given by

$$\tilde{\nabla}^2 \rho(\mathbf{r}, t) \equiv \frac{1}{r^{\mathcal{N}_f-1}} \frac{\partial}{\partial r} \left(r^{\mathcal{N}_f-1} \frac{\partial}{\partial r} \rho(\mathbf{r}, t) \right) + \frac{1}{r^2 \sin^{\mathcal{N}_f-2} \theta} \frac{\partial}{\partial \theta} \left(\sin^{\mathcal{N}_f-2} \theta \frac{\partial}{\partial \theta} \rho(\mathbf{r}, t) \right), \quad (2)$$

where \mathcal{N}_f represents a noninteger dimension [44]. It is interesting to note that the presence of the noninteger dimension has also been considered to investigate the electromagnetic wave propagation in fractional space [50]. The propagator for Eq. (1) is first investigated by considering the system subjected to the boundary conditions $\rho(\mathbf{r}, t)|_{r=a} = \Phi(\theta, t)$ and $\partial_r \rho(\mathbf{r}, t)|_{r=0} = 0$. Next, we extend our analysis to the case $\lim_{r \rightarrow \infty} \rho(\mathbf{r}, t) = 0$. In both cases, we first obtain the solutions in the absence of the nonlocal term and later, we analyze the solutions by incorporating the nonlocal term where the kernel $\alpha(t)$ quantifies how the quantity $\rho(\mathbf{r}, t)$ is influenced by $\rho(\mathbf{r}, \bar{t})$. At this point, it is interesting to note that Eq. (1) has a particular case where the situations work out in Refs. [51–53]. These developments are performed in Section 2, while in Section 3, the discussions and conclusions are presented.

2. Fractional equation

Let us start our discussion by considering the previous diffusion equation in the absence of the nonlocal term and subjected to the boundary conditions $\partial_r \rho(\mathbf{r}, t)|_{r=0} = 0$ and $\rho(\mathbf{r}, t)|_{r=a} = \Phi(\theta, t)$. After that, we incorporate the nonlocal term and extend the result to the case $\lim_{r \rightarrow \infty} \rho(r, t) = 0$. For these cases, the initial condition is an arbitrary normalized distribution function in the interval considered. The diffusion equation to be considered is given by

$$\frac{\partial^\gamma}{\partial t^\gamma} \rho(\mathbf{r}, t) = \mathcal{D} \tilde{\nabla}^2 \rho(\mathbf{r}, t). \quad (3)$$

This equation has the usual diffusion equation for $\gamma = 1$ and \mathcal{N}_f integer as a particular case. In order to solve the equation accomplishing the required boundary condition, we use the Laplace transform and the Green function approach. By performing the Laplace transform, we obtain

$$\mathcal{D} s^{1-\gamma} \tilde{\nabla}^2 \rho(\mathbf{r}, t) - s \rho(\mathbf{r}, s) = -\rho(\mathbf{r}, 0) \quad (4)$$

where the last term on the right side represents an arbitrary normalized initial condition. The formal solution for this equation is given by

$$\rho(\mathbf{r}', s) = - \int_0^a dr r^{\mathcal{N}_f-1} \int_0^\pi d\theta \sin^{\mathcal{N}_f-2} \theta \rho(\mathbf{r}', 0) \mathcal{G}(\mathbf{r}, \mathbf{r}'; s) - a^{\mathcal{N}_f-1} \int_0^\pi d\theta \sin^{\mathcal{N}_f-2} \theta \Phi(\theta, s) \frac{\partial}{\partial r} \mathcal{G}(\mathbf{r}, \mathbf{r}'; s) \Big|_{r=a}, \quad (5)$$

with the propagator (or the Green function) determined by the equation

$$\mathcal{D} s^{1-\gamma} \tilde{\nabla}^2 \mathcal{G}(\mathbf{r}, \mathbf{r}'; s) - s \mathcal{G}(\mathbf{r}, \mathbf{r}'; s) = \frac{1}{r^{\mathcal{N}_f-1} \sin^{\mathcal{N}_f-2} \theta} \delta(r - r') \delta(\theta - \theta') \quad (6)$$

and subjected to the conditions $\partial_r \mathcal{G}(\mathbf{r}, \mathbf{r}'; s)|_{r=0} = 0$ and $\mathcal{G}(\mathbf{r}, \mathbf{r}'; s)|_{r=a} = 0$. Note that the last term of Eq. (5) represents the surface effects (e.g., adsorption–desorption processes) on the relaxation of the system's initial condition.

In order to obtain the solution for Eq. (6), we may use the eigenfunctions of the Sturm–Liouville problem related to the spatial operator, i.e., one solves the equation $\tilde{\nabla}^2 \psi = -k^2 \psi$, with ψ satisfying the boundary conditions mentioned above for $\mathcal{G}(\mathbf{r}, \mathbf{r}'; s)$. In this sense, it is possible to show that

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; s) = \sum_{n=1, l=0}^{\infty} \mathcal{A}_{n,l}(\mathbf{r}', s) \psi(\mathbf{r}, k_{n,l}) \quad (7)$$

with the eigenfunction given by

$$\psi(\mathbf{r}, k_{n,l}) = \mathcal{N}_{n,l} r^{1-\alpha/2} J_\nu(k_{n,l} r) C_l^{\alpha/2-1}(\cos \theta) \quad (8)$$

where $J_\nu(k_{n,l} r)$ is the Bessel function with $\nu = \alpha/2 + l - 1$, $C_l^{\alpha/2-1}(\cos \theta)$ are the Gegenbauer polynomials [54] with

$$\mathcal{A}_{n,l}(\mathbf{r}', s) = \frac{\psi(\mathbf{r}', k_{n,l})}{s + \mathcal{D} s^{1-\gamma} k_{n,l}^2}, \quad (9)$$

$$\mathcal{N}_{n,l}^2 = \frac{2}{a^2 J_{\nu+1}^2(k_{n,l} a)} \frac{l! \left(l + \frac{\mathcal{N}_f}{2} - 1 \right)}{2^{3-\mathcal{N}_f} \pi \Gamma(l + \mathcal{N}_f - 2)} \left[\Gamma\left(\frac{\mathcal{N}_f}{2} - 1 \right) \right]^2, \quad (10)$$

and the eigenvalues are determined by the equation $J_\nu(k_{n,l}a) = 0$. Note that the solutions for Eq. (6) may also be found by using the procedure presented in Ref. [55] based on the finite Hankel transforms. By performing the inverse Laplace transform, we obtain that

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; t) = \sum_{n=1, l=0}^{\infty} \psi(\mathbf{r}', k_{n,l}) \psi(\mathbf{r}, k_{n,l}) E_\gamma(-\mathcal{D}k_{n,l}^2 t^\gamma). \quad (11)$$

The presence of the Mittag-Leffler function [49] in the Green function indicates an anomalous relaxation of the system which, for lesser times, is governed by a stretched exponential, i.e., $E_\gamma(x) \sim e^{-x^\gamma/\Gamma(1+\gamma)}$, and, for longer times, a power-law behavior, i.e., $E_\gamma(x) \sim -1/(\Gamma(1-\gamma)x)$, which dominates the relaxation process.

In the presence of the nonlocal term, the solution of Eq. (1) is formally given by Eq. (5); however, the Green function for this case is given by

$$\mathcal{G}_{\text{nonlocal}}(\mathbf{r}, \mathbf{r}'; t) = \mathcal{G}(\mathbf{r}, \mathbf{r}'; t) + \sum_{n=1, l=0}^{\infty} \psi(\mathbf{r}', k_{n,l}) \psi(\mathbf{r}, k_{n,l}) \Phi(t, k_{n,l}), \quad (12)$$

with

$$\Phi(t, k_{n,l}) = \sum_{k=1}^{\infty} \frac{(-1)^k}{\Gamma(1+k)} \int_0^t dt_n \alpha(t-t_n) \int_0^{t_n} dt_{n-1} \alpha(t_n-t_{n-1}) \cdots \int_0^{t_2} dt_1 \alpha(t_2-t_1) t_1^{\gamma n} E_\gamma^{(n)}(-\mathcal{D}k_{n,l}^2 t^\gamma). \quad (13)$$

Let us extend the previous results by considering the unconfined region, i.e., a free diffusion, where the surface effects are not relevant. In order to investigate this scenario, we first consider, for simplicity, the situation characterized by radial symmetry and, then, the angular dependence is incorporated. In the first scenario, the system only depends on the variable r . One may use the eigenfunctions of the Sturm–Liouville problem related to the spatial operator as in the previous case, with the same boundary conditions for ρ . For this case, one finds

$$\mathcal{G}(r, r'; s) = \int_0^\infty dk k \varphi(r, k) \bar{\mathcal{G}}(k, r'; s), \quad (14)$$

whose inverse is given by

$$\bar{\mathcal{G}}(k, r'; s) = \int_0^\infty dr r^{\alpha-1} \varphi(r, k) \mathcal{G}(r, r'; s), \quad (15)$$

where $\varphi(r, k) = r^{1-\alpha/2} J_\nu(kr)$, $\nu = \mathcal{N}_f/2 - 1$, and $J_\nu(x)$ is the Bessel function. It is possible to show that

$$\bar{\mathcal{G}}(k, r'; s) = \frac{\varphi(r', k)}{s + \mathcal{D}s^{1-\gamma} k^2}. \quad (16)$$

The inverse Laplace transform of Eq. (16) yields

$$\bar{\mathcal{G}}(k, r'; t) = \varphi(r', k) E_\gamma(-\mathcal{D}k^2 t^\gamma). \quad (17)$$

Note the presence of the Mittag-Leffler function [49] in the previous equation which implies an anomalous spreading of the initial condition as the case worked out before. By using the previous results, one finds that the propagator is given by

$$\mathcal{G}(r, r'; t) = \int_0^\infty dk k \varphi(r', k) \varphi(r, k) E_\gamma(-\mathcal{D}k^2 t^\gamma). \quad (18)$$

In particular, it can be written as

$$\mathcal{G}(r, r'; t) = \frac{rr'}{(rr')^{\mathcal{N}_f/2}} H_{2, [0:1], 0, [0:2]}^{1, 0, 1, 1, 1} \left[\begin{matrix} (r'/r)^2 \\ \mathcal{D}t^\gamma/r^2 \end{matrix} \left| \begin{matrix} \left(\frac{2-\nu}{2}, 1\right); \left(\frac{2+\nu}{2}, 1\right) \\ -; (0, 1) \\ \left(-\frac{\nu}{2}, 1\right); \left(\frac{\nu}{2}, 1\right); (0, 1), (0, \gamma) \end{matrix} \right. \right], \quad (19)$$

where

$$H_{E, [A:C], F, [B, D]}^{L, M, M_1, N, N_1} \left[\begin{matrix} x \\ y \end{matrix} \left| \begin{matrix} (\varepsilon_1, \omega_1), \dots, (\varepsilon_E, \omega_E) \\ (a_1, \alpha_1), \dots, (a_A, \alpha_A); (c_1, \gamma_1), \dots, (c_C, \gamma_C) \\ (\xi_1, \varpi_1), \dots, (\xi_F, \varpi_F) \\ (b_1, \beta_1), \dots, (b_B, \beta_B); (d_1, \delta_1), \dots, (d_D, \delta_D) \end{matrix} \right. \right] \quad (20)$$

is the generalized Fox H function [56,57]. For the particular case $\gamma = 1$, Eq. (18) can be simplified to

$$\mathcal{G}(r, r'; t) = \frac{1}{2\mathcal{D}t} (rr')^{1-\mathcal{N}_f/2} e^{-\frac{1}{4\mathcal{D}t}(r^2+r'^2)} I_\nu\left(\frac{rr'}{2\mathcal{D}t}\right). \quad (21)$$

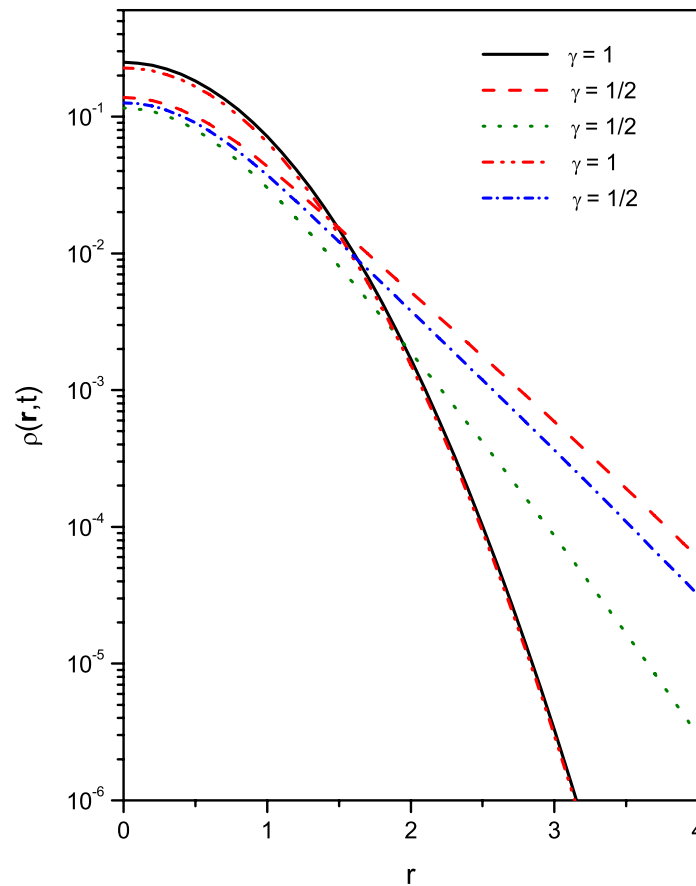


Fig. 1. The behavior of $\rho(\mathbf{r}, t)$ versus r is illustrated by considering different time dependences for the nonlocal term. The red dashed–dotted line corresponds to the usual case, i.e., $\gamma = 1$ with $\alpha(t) = \alpha\delta(t)$. The black solid line is the usual diffusion equation in the presence of the nonlocal term with $\alpha(t) = \alpha e^{-\eta t}$. The blue short dashed–dotted line corresponds to the fractional case with $\alpha(t) = \alpha\delta(t)$. The red dashed line and the green dotted line are the fractional case with $\alpha(t) = \alpha t^{\mu-1}/\Gamma(\mu)$ with $\mu = 1/2$ and $\mu = -1/2$. For simplicity, $\mathcal{D} = 1$, $\xi = 0.1$, $\alpha = 1$, $t = 0.1$, and $\mathcal{N}_f = 3$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

By incorporating the angular variable in previous calculations, the propagator becomes

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; t) = \sum_{l=0}^{\infty} \int_0^{\infty} dp p \Theta_l(\theta) \Theta_l(\theta') \varphi_l(r', p) \varphi_l(r, p) E_{\gamma}(-\mathcal{D}p^2 t^{\gamma}), \tag{22}$$

with

$$\Theta_l(\theta) = \mathcal{N}_l C_l^{\mathcal{N}_f/2-1}(\cos \theta), \tag{23}$$

where $C_l^{\alpha}(\cos \theta)$ are the Gegenbauer polynomials [54], and

$$\mathcal{N}_l^2 = \frac{l! \left(l + \frac{\mathcal{N}_f}{2} - 1\right)}{2^{3-\mathcal{N}_f} \pi \Gamma(l + \mathcal{N}_f - 2)} \left[\Gamma\left(\frac{\mathcal{N}_f}{2} - 1\right)\right]^2. \tag{24}$$

The functions $\varphi_l(r, p)$ are defined as before, with $\nu = \alpha/2 + l - 1$. The solution of Eq. (5), after inverting the Laplace transform, can be written in terms of the Green function as follows:

$$\rho(\mathbf{r}', t) = - \int_0^{\infty} dr r^{\mathcal{N}_f-1} \int_0^{\pi} d\theta \sin^{\mathcal{N}_f-2} \theta \rho(\mathbf{r}', 0) \mathcal{G}(\mathbf{r}, \mathbf{r}'; t). \tag{25}$$

Similar to the previous case, the presence of the nonlocal term changes only the time behavior of the Green function. In fact, after performing some calculations, it is possible to show that

$$\mathcal{G}_{\text{nonlocal}}(\mathbf{r}, \mathbf{r}'; t) = \mathcal{G}(\mathbf{r}, \mathbf{r}'; t) + \sum_{l=0}^{\infty} \int_0^{\infty} dp p \Theta_l(\theta) \Theta_l(\theta') \varphi_l(r', p) \varphi_l(r, p) \Phi(t, p), \tag{26}$$

with $\Phi(t, p)$ defined by Eq. (13), where $k_{n,l}$ is replaced by p . By using Eq. (25), we illustrate in Fig. 1 the influence of the fractional derivative and the nonlocal term on the spreading of the initial condition $\rho(\mathbf{r}, 0) \propto e^{-\xi r^2}$. Note that, for the case $\gamma \neq 1$, the asymptotic behavior is different from the usual one. In Fig. 2, we show the behavior of the second moment for the initial condition $\rho(\mathbf{r}, 0) \propto \delta(\mathbf{r})$.

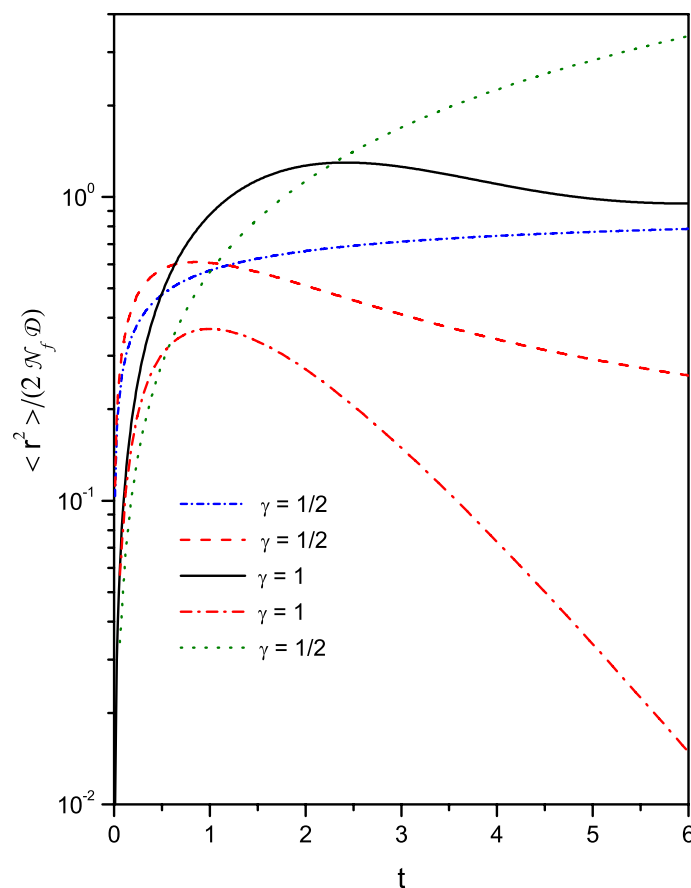


Fig. 2. The behavior of $\langle r^2 \rangle / (2\mathcal{N}_f \mathcal{D})$ versus t is illustrated for different values of γ by considering the same time dependences for the nonlocal term presented in Fig. 1. For simplicity, $\mathcal{D} = 1$ and $\alpha = 1$.

3. Discussions and conclusions

We investigate the solutions of a fractional diffusion with noninteger dimensions in the presence of a nonlocal term, by first considering the system in a confined region with inhomogeneous time dependent boundary conditions. Next, we extend the previous solution for an unlimited region. In both cases, the solutions present an anomalous spreading of the initial condition which may be connected to anomalous diffusion. In Figs. 1 and 2, we have illustrated the spreading of a system initially characterized by a Gaussian distribution and the behavior of the second moment when the initial condition of the system is a delta function. These figures show that the system's behavior for both initial conditions is different from the usual. Finally, we hope that the results found here can be useful for discussing situations connected to anomalous diffusion.

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