JOURNAL OF MATHEMATICAL PHYSICS **54**, 082107 (2013)

Time dependent solutions for a fractional Schrödinger equation with delta potentials

E. K. Lenzi,[1] H. V. Ribeiro,[1] M. A. F. dos Santos,[1] R. Rossato,[2] and R. S. Mendes[1]

[1]*Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá – PR, Brazil*

[2]*Universidade Tecnológica Federal do Paraná - Câmpus Apucarana, Rua Marcílio Dias 635, 86812 - 460 Apucarana - PR, Brazil*

(Received 8 May 2013; accepted 12 August 2013; published online 30 August 2013)

We investigate, for an arbitrary initial condition, the time dependent solutions for a fractional Schrödinger equation in the presence of delta potentials by using the Green function approach. The solutions obtained show an anomalous spreading asymptotically characterized by a power-law behavior, which is governed by the order of the fractional spatial operator present in the Schrödinger equation. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4819253]

I. INTRODUCTION

The idea of fractional dimension introduced by Hausdorff has attracted the attention of several researchers and became widely used after the pioneers works of Mandelbrot[1] about the fractal nature of several phenomena in many fields of science. In this context, the concept of noninteger order differentiation (fractional calculus) that emerges from the works of, for example, Leibniz, Liouville, Riemann, Grunwald, Caputo, and Letnikov[2–5] has an effervescent scenario characterized by a continuously increasing list of applications from nonlinear dynamics[6] to anomalous transport.[7–13] An interesting aspect of this formalism concerns to the nonlocal and non-Markovian effects, which can be elegantly incorporated in the evolution equations by a suitable extension of the differential operators of integer order to operators of noninteger orders. In connection with this discussion, one of the relevant problems is how to extend the Schrödinger equation in order to incorporated nonlocal and non-Markovian characteristics. This point was worked out in Refs. 14–17 by using an extension of the Feymann path integral and investigated in Ref. 18 by incorporating fractional operators in the Schrödinger equation. Other situations about the fractional Schrödinger equation that emerges from these developments have been extensively investigated and analyzed in different contexts.[19–29] Following these developments, our goal is to obtain time dependent solutions, i.e., the Green functions (or propagator) which play an important role in this scenario of a fractional Schrödinger equation under the presence of delta potentials. This development is performed in Sec. II where we investigate the fractional Schrödinger equation by considering the potentials: (i) $V(x) = \mathcal{V}\delta(x)$ and (ii) $V(x) = \mathcal{V}_1\delta(x - l_1) + \mathcal{V}_2\delta(x - l_2)$. The first case corresponds to a delta potential at the origin and in the second case the potential is composed of two delta functions at the positions l_1 and l_2. In Sec. III, we present our discussion and conclusions.

II. FRACTIONAL SCHRÖDINGER EQUATION

Let us start our investigation about the time dependent solutions by considering the fractional Schrödinger equation[16,17] in the presence of the potential $V(x) = \mathcal{V}\delta(x)$ and subjected to the boundary condition $\Psi(\pm\infty, t) = 0$ with an arbitrary initial condition $\Psi(x, 0) = \Phi(x)$, where $\Phi(x)$ is an arbitrary function. For this case, the fractional Schrödinger equation can be written as

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_\mu\left(-\hbar^2\nabla^2\right)^{\frac{\mu}{2}}\Psi(x,t) + \mathcal{V}\delta(x)\Psi(x,t)\,, \tag{1}$$

where $\mathcal{D}_\mu$ is a constant and the spatial operator in the Fourier space ($\mathcal{F}\{\Psi(x,t)\} = \int_{-\infty}^{\infty} dx\, e^{-i\frac{p}{\hbar}x}\Psi(x,t) = \overline{\Psi}(p,t)$ and $\mathcal{F}^{-1}\{\overline{\Psi}(p,t)\} = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp\, e^{i\frac{p}{\hbar}x}\overline{\Psi}(p,t) = \Psi(x,t)$) is given by

$$\mathcal{F}\left\{\left(-\hbar^2\nabla^2\right)^{\frac{\mu}{2}}\Psi(x,t)\right\} = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{-i\frac{p}{\hbar}x}|p|^{\mu}\overline{\Psi}(p,t)\,, \tag{2}$$

which corresponds a Riesz-Welly operator.[2] The usual case is recovered from the previous development for $\mu = 2$ and, consequently, by choosing $\mathcal{D}_{\mu=2} = \hbar^2/(2m)$. The discussion involving the presence of fractional time derivative will be performed later on, after the development of the solution for the case characterized by two delta function, i.e., the case (ii).

The solutions and properties of Eq. (1) has been investigated in several contexts (see, for example, Refs. 28–30). Here, we focus our attention on a different point: the time dependent solutions and, consequently, the influence of the spatial fractional derivative on the spreading of the solution by using the Green function approach. In this framework, the solution of Eq. (1) can be written as

$$\Psi(x,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' \mathcal{G}(x,x';t,t')\Phi(x)\,, \tag{3}$$

with the Green function governed by the following equation

$$i\hbar\frac{\partial}{\partial t}\mathcal{G}(x,x';t,t') - \mathcal{H}\mathcal{G}(x,x';t,t') = i\hbar\delta(x-x')\delta(t-t'), \tag{4}$$

where

$$\mathcal{H}\mathcal{G}(x,x';t,t') = \mathcal{D}_\mu\left(-\hbar^2\nabla^2\right)^{\frac{\mu}{2}}\mathcal{G}(x,x';t,t') + \mathcal{V}\delta(x)\mathcal{G}(x,x';t,t') \tag{5}$$

with $\mathcal{G}(\pm\infty,x';t,t') = 0$ and $\mathcal{G}(x,x';t,t') = 0$ for $t < t'$ (causality condition). Applying the Laplace ($\mathcal{L}\{\Psi(x,t)\} = \int_0^{\infty} dt\, e^{-st}\Psi(x,t) = \widetilde{\Psi}(x,s)$ and $\mathcal{L}^{-1}\{\widetilde{\Psi}(x,s)\} = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st}\widetilde{\Psi}(x,s) = \Psi(x,t)$) and Fourier transforms in Eq. (4), we obtain that

$$\widetilde{\overline{\mathcal{G}}}(p,x';s,t') = \widetilde{\overline{\mathcal{G}}}_f(p,s)e^{-i\frac{p}{\hbar}x'}e^{-st'} + \mathcal{V}\widetilde{\overline{\mathcal{G}}}_f(p,s)\widetilde{\mathcal{G}}(0,x';s,t'), \tag{6}$$

where

$$\widetilde{\overline{\mathcal{G}}}_f(p,s) = \frac{1}{s + i\mathcal{D}_\mu|p|^{\mu}/\hbar}\,. \tag{7}$$

Equation (7) corresponds to the Green function of the cases analyzed in Refs. 14–17, where the fractional Schrödinger equation is considered in absence of potential term, i.e., $\mathcal{V}(x,t) = 0$, in a nonlimited spatial region, and for $\mu = 2$ it recovers the usual form of the propagator in the Laplace-Fourier space. Performing some calculations, it is possible to show that

$$\widetilde{\mathcal{G}}(0,x';s,t') = \frac{e^{-st'}}{1-\mathcal{V}\widetilde{\mathcal{G}}_f(0,s)}\widetilde{\mathcal{G}}_f(x',s), \tag{8}$$

$$\widetilde{\mathcal{G}}_f(0,s) = \frac{1}{\mu}\left(\frac{\hbar}{i\mathcal{D}_\mu}\right)^{\frac{1}{\mu}}\frac{s^{\frac{1}{\mu}-1}}{\sin(\pi/\mu)}\,. \tag{9}$$

By substituting Eqs. (7)–(9) in Eq. (6) and performing the inverse Laplace and Fourier transforms, we obtain that

$$\begin{aligned}\mathcal{G}(x,x';t,t') &= \mathcal{G}_f(x-x',t)\theta(t-t')\\ &+ \mathcal{V}\theta(t-t')\int_0^t d\eta\mathcal{G}_f(x,t-\eta)\left(\mathcal{G}_f(x',\eta) + \int_0^{\eta} d\xi\,\Delta_{\mathcal{V}}(\eta-\xi)\mathcal{G}_f(x',\xi)\right)\end{aligned} \tag{10}$$

with

$$\mathcal{G}_f(x,t) = \frac{1}{\mu|x|}\,\mathrm{H}^{1,1}_{2,2}\left[\frac{|x|}{\left(\mathcal{D}_\mu it/\hbar\right)^{\frac{1}{\mu}}}\middle|\begin{matrix}\left(1,\frac{1}{\mu}\right) & \left(1,\frac{1}{2}\right)\\ (1,1) & \left(1,\frac{1}{2}\right)\end{matrix}\right] \tag{11}$$

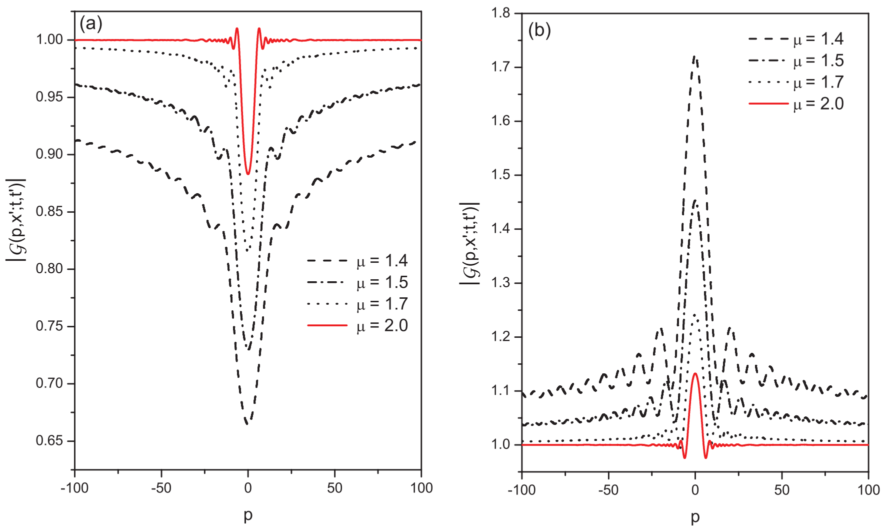

FIG. 1. Figures 1(a) and 1(b) illustrate the behavior of Eq. (10) for the cases repulsive ($\mathcal{V} = 1$) and attractive ($\mathcal{V} = -1$) by considering different values of μ and, for simplicity, $x' = 0$, $t = 0.1$, $\hbar = 1$, and $\mathcal{D}_\mu = 1$.

and

$$\Delta_{\mathcal{V}}(t) = \mathcal{V}\mathcal{E}t^{-1/\mu}\mathrm{E}_{1-1/\mu,1-1/\mu}\left(\mathcal{V}\mathcal{E}t^{1-1/\mu}\right), \tag{12}$$

$$\mathcal{E} = \frac{1}{\mu}\left(\frac{\hbar}{i\mathcal{D}_\mu}\right)^{\frac{1}{\mu}}\frac{1}{\sin(\pi/\mu)}\,, \tag{13}$$

where $\mathrm{E}_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function.[31,32] The Fox H-function (see the Appendix for some properties) present in Eq. (11) is asymptotically governed by a power-law behavior, in contrast to the usual case that is characterized by an exponential relaxation. In fact, it is possible to show that $\mathcal{G}_f(x,t) \sim i\mathcal{D}_\mu t/(\mu\hbar|x|^{1+\mu})$, for $\mu \neq 1$. The generalized Mittag-Leffler function, in Eq. (12), has also a power-law behavior in the asymptotic limit of long times. These characteristics incorporated in Eq. (10) lead us to a different behavior from the usual case. In Figs. 1(a) and 1(b), we illustrate the behavior of Eq. (10) for a repulsive and an attractive potential, i.e., $\mathcal{V} > 0$ and $\mathcal{V} < 0$. In the first case, the particle experiences an ultrathin barrier and in the second case there is one bound state. Note that the influence of the potential increases for values of $\mu \to 1$ and decrease for μ closed to the usual case, i.e., $\mu \to 2$. This feature is connected with the asymptotic behavior manifested by the Green function for $x \to \infty$, which has a long tailed behavior for $\mu \to 1$ and has a short tailed behavior, similarly to the usual case, for $\mu \to 2$. In addition, Eq. (10) extends results presented for the standard case in Refs. 33 and 34 to a fractional Schrödinger equation.

Now, we extend the previous development by considering the fractional Schrödinger equation in the presence of the potential $V(x) = \mathcal{V}_1\delta(x - l_1) + \mathcal{V}_2\delta(x - l_2)$. This potential is characterized by two delta functions, one at the point $x = l_1$ and the other at the point $x = l_2$. Note that this double barrier structure is the electronic analog to a Fabry–Perot interferometer. In addition, a double barrier structure may be used to approximate several configurations such as the related to resonant tunneling in semiconductor quantum-well structures or in quantum transport, where the spatial separation of the barriers is large compared to the individual barrier thickness.[33]

By substituting this potential in Eq. (1), we obtain

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_\mu\left(-\hbar^2\nabla^2\right)^{\frac{\mu}{2}}\Psi(x,t) + \mathcal{V}_1\delta(x-l_1)\Psi(x,t) + \mathcal{V}_2\delta(x-l_2)\Psi(x,t). \tag{14}$$

In order to solve this equation, we also use Green function approach and Laplace-Fourier transforms as performed for the previous case. From Eq. (14), we obtain that

$$\widetilde{\widetilde{\mathcal{G}}}(p,x';s,t') = \widetilde{\widetilde{\mathcal{G}}}_f(p,s)e^{-i\frac{p}{\hbar}x'}e^{-st'} + \mathcal{V}_1 e^{-i\frac{p}{\hbar}l_1}\widetilde{\mathcal{G}}(l_1,x';s,t')\widetilde{\widetilde{\mathcal{G}}}_f(p,s) + \mathcal{V}_2 e^{-i\frac{p}{\hbar}l_2}\widetilde{\mathcal{G}}(l_2,x';s,t')\widetilde{\widetilde{\mathcal{G}}}_f(p,s)\,. \tag{15}$$

Applying the inverse Fourier and Laplace transforms in Eq. (15), it is possible to show that

$$\mathcal{G}(x,x';t,t') = \mathcal{G}_f(x-x',t)\theta(t-t') + \mathcal{V}_1\int_0^t d\bar{t}\mathcal{G}_f(x-l_1,t-\bar{t})\mathcal{G}(l_1,x';\bar{t},t') + \mathcal{V}_2\int_0^t d\bar{t}\mathcal{G}_f(x-l_2,t-\bar{t})\mathcal{G}(l_2,x';\bar{t},t')\,. \tag{16}$$

The first term of Eq. (16) corresponds to the Green function of the free case and the other terms give the effect of the potential on the first term. In order to formally determine the previous Green function, we have to find the functions $\mathcal{G}(l_1,x';t,t')$ and $\mathcal{G}(l_2,x';t,t')$. After some calculations, it is possible to show that they can be written as

$$\mathcal{G}(l_1,x';t,t') = \theta(t-t')\int_0^t d\vartheta\,\Upsilon(t-\xi)\Bigg\{\mathcal{G}_f(l_1-x',\xi) + \mathcal{V}_2\int_0^\xi d\eta\Big[\mathcal{G}_f(l_1-l_2,\xi-\eta)\mathcal{G}_f(l_2-x',\eta) - \mathcal{G}_f(0,\vartheta-\eta)\mathcal{G}_f(l_1-x',\eta)\Big]\Bigg\} \tag{17}$$

and

$$\mathcal{G}(l_2,x';t,t') = \theta(t-t')\int_0^t d\xi\,\Upsilon(t-\xi)\Bigg\{\mathcal{G}_f(l_1-x',\xi) + \mathcal{V}_1\int_0^\xi d\eta\Big[\mathcal{G}_f(l_1-l_2,\xi-\eta)\mathcal{G}_f(l_1-x',\eta) - \mathcal{G}_f(0,\vartheta-\eta)\mathcal{G}_f(l_2-x',\eta)\Big]\Bigg\}, \tag{18}$$

where

$$\Upsilon(t) = \Xi(t) + \sum_{n=1}^{\infty}(\mathcal{V}_1\mathcal{V}_2)^n\int_0^t dt_n\Theta(t-t_n)\cdots\int_0^{t_2}dt_1\Theta(t_2-t_1)\int_0^{t_1}d\xi\Theta(t_1-\xi)\Xi(\xi) \tag{19}$$

with

$$\Theta(t) = \int_0^t d\zeta\,\Lambda_{\mathcal{V}_1}(t-\zeta)\Lambda_{\mathcal{V}_2}(\zeta), \tag{20}$$

$$\Lambda_{\mathcal{V}_1,\mathcal{V}_2} = \mathcal{G}_f(|l_1-l_2|,t) + \int_0^t d\xi\,\Delta_{\mathcal{V}_1,\mathcal{V}_2}(\xi)\mathcal{G}_f(|l_1-l_2|,t-\xi), \tag{21}$$

and

$$\Xi(t) = \delta(t) + \frac{\mathcal{V}_2^2 t^{-\frac{1}{\mu}}}{\mathcal{V}_2-\mathcal{V}_1}\mathcal{E}\mathrm{E}_{1-\frac{1}{\mu},1-\frac{1}{\mu}}\left(\mathcal{V}_2\mathcal{E}t^{1-\frac{1}{\mu}}\right) - \frac{\mathcal{V}_1^2 t^{-\frac{1}{\mu}}}{\mathcal{V}_2-\mathcal{V}_1}\mathcal{E}\mathrm{E}_{1-\frac{1}{\mu},1-\frac{1}{\mu}}\left(\mathcal{V}_1\mathcal{E}t^{1-\frac{1}{\mu}}\right)\,. \tag{22}$$

The result obtained for the last case extends, for the fractional case, the result obtained in Ref. 35 which, as discussed for the first case initially presented, is characterized by a different relaxation process from the usual case.

Let us discuss the changes produced on the solution when fractional time derivatives are incorporated in above equations.[36] For simplicity, we focus our analysis on Eq. (1), without loss of generality. In this case, the solution for this equation, when the usual time derivative is replaced by a fractional time derivative in the Caputo sense,[2] is given by

$$\Psi(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t\frac{d\bar{t}}{(t-\bar{t})^\gamma}\int_{-\infty}^{\infty}dx'\mathcal{G}_\gamma(x,x';\bar{t},t')\Phi(x')\,. \tag{23}$$

The Green function is obtained by solving the following equation

$$i\hbar\frac{\partial^{\gamma}}{\partial t^{\gamma}}\mathcal{G}_{\gamma}(x,x';t,t')-\mathcal{H}\mathcal{G}_{\gamma}(x,x';t,t')=i\hbar\delta(x-x')\delta(t-t') \tag{24}$$

with $0<\gamma<1$, subjected to the boundary conditions $\mathcal{G}(\pm\infty,x';t,t')=0$, and the fractional time derivative[2] defined as

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}}\mathcal{G}_{\gamma}(x,x';t,t')=\frac{1}{\Gamma(n-\gamma)}\int_0^t\frac{d\bar{t}}{(t-\bar{t})^{\gamma+1-n}}\mathcal{G}_{\gamma}^{(n)}(x,x';\bar{t},t')\,, \tag{25}$$

where $n-1<\gamma<n$ and $\mathcal{G}_{\gamma}^{(n)}(x,x';t,t')=\partial_t^n\mathcal{G}_{\gamma}(x,x';t,t')$. Employing the previous procedure of calculations, it is possible to show that the solution of Eq. (24) in the Fourier space is given by

$$\mathcal{G}_{\gamma}(p,x';t,t')=\overline{\mathcal{G}}_{f,\gamma}(p,t)e^{-i\frac{p}{\hbar}x'}\theta(t-t')+\mathcal{V}\int_0^t\overline{\mathcal{G}}_{f,\gamma}(p,\bar{t})\widetilde{\mathcal{G}}_{\gamma}(0,x';t-\bar{t},t')d\bar{t} \tag{26}$$

with $\overline{\mathcal{G}}_{f,\gamma}(p,t)=t^{\gamma-1}\mathrm{E}_{\gamma,\gamma}(-i\mathcal{D}_{\mu}|p|^{\mu}t^{\gamma}/\hbar)$,

$$\mathcal{G}_{\gamma}(0,x';t,t')=\mathcal{G}_{f,\gamma}(x',t)\theta(t-t')+\theta(t-t')\int_0^t\mathcal{G}_{f,\gamma}(x',\eta)\Delta_{\mathcal{V},\gamma}(t-\eta)d\eta \quad \text{and} \tag{27}$$

$$\Delta_{\mathcal{V},\gamma}(t)=\mathcal{V}\mathcal{E}t^{(1-1/\mu)\gamma-1}\mathrm{E}_{\gamma-\gamma/\mu,\gamma-\gamma/\mu}\left(\mathcal{V}\mathcal{E}t^{\gamma-\gamma/\mu}\right).$$

At this point, it is interesting to observe the presence of the generalized Mittag-Leffer function in the solution. This function has a power law behavior for $\overline{\mathcal{G}}_{f,\gamma}(p,t)$ in the asymptotic limit of $|p|\to\infty$. In contrast, the previous case is governed by a stretched exponential in the Fourier space, which changes the behavior of the solutions. By performing the inverse of Fourier transform in Eq. (26), we obtain

$$\mathcal{G}_{\gamma}(x,x';t,t')=\mathcal{G}_{f,\gamma}(x-x',t)\theta(t-t')+\mathcal{V}\int_0^t\mathcal{G}_{f,\gamma}(x,\bar{t})\mathcal{G}_{\gamma}(0,x';t-\bar{t},t')d\bar{t}$$

$$\mathcal{G}_{f,\gamma}(x,t)=\frac{1}{\mu|x|}\mathrm{H}_{3,3}^{2,1}\left[\frac{|x|}{(\mathcal{D}_{\mu}it/\hbar)^{\frac{1}{\mu}}}\middle|\begin{matrix}(1,1) & \left(\gamma,\frac{\gamma}{\mu}\right) & \left(1,\frac{1}{2}\right)\\ \left(1,\frac{1}{\mu}\right) & \left(1,\frac{1}{\mu}\right) & \left(1,\frac{1}{2}\right)\end{matrix}\right]. \tag{28}$$

Figure 2(a) shows the solution given by Eq. (23) for $\gamma=1/2$ and for different values of μ, and Fig. 2(b) illustrates the behavior of the Green function given by Eq. (26) for different times for $\gamma=1/2$ with $\mu=1.5$. Note that, differently from the previous results obtained for $\gamma=1$ (see Fig. 1 where $|\mathcal{G}(p,x';t,t')|\to 1$ to $|p|\to\infty$), the case $\gamma\neq 1$ lead us to different asymptotic behavior for the Green function. This feature is connected to the presence of the generalized Mittag-Leffler function in the solution of the free case, which introduces a nonexponential asymptotic behavior for the solution.

III. DISCUSSION AND CONCLUSIONS

We have investigated the solutions of a fractional Schrödinger equation in the presence of the delta potentials. We have first considered the fractional Schrödinger equation in the presence of a single delta potential. For this case, we have obtained the time dependent solution for an arbitrary initial condition in terms of the Green function approach. In Figs. 1(a) and 1(b), we have illustrated, for this case, the behavior of the Green function by considering different values of μ and shown that the influence of the effect of potential on the time evolution is greater for μ values closed one. This last point is connected with the long tailed behavior of the solution imposed by the spatial fractional derivative that depends on the index μ. For the second case, we have also obtained the time dependent solution in terms of the Green function approach and, similarly to the first case, a different relaxation process from the usual one is evidenced by the solutions. After these analysis, we have investigated the effect obtained on the solution when the usual time derivative is replaced by a fractional time derivative. For simplicity, we have worked out the situation characterized by a single delta potential and shown that the solution exhibited a different behavior of the one obtained

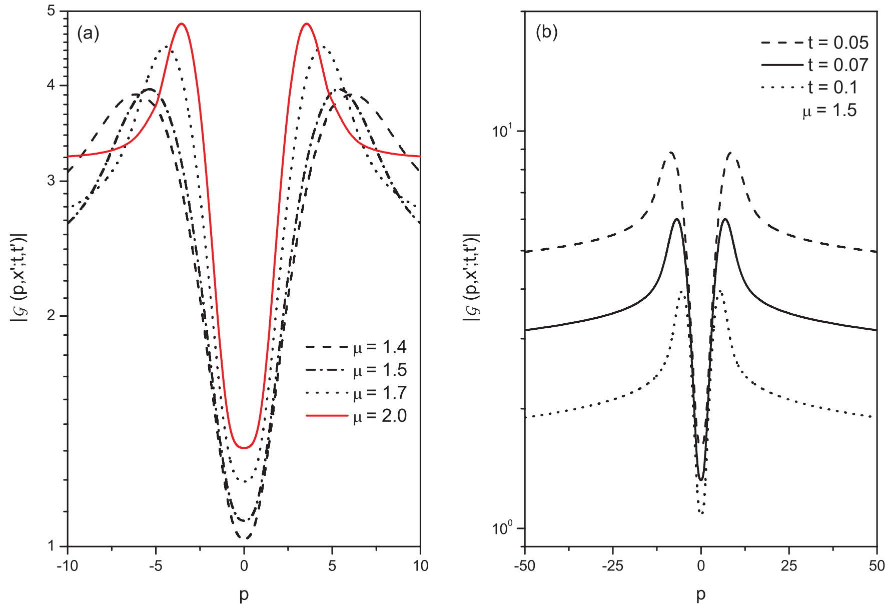

FIG. 2. Figures 2(a) and 2(b) illustrate the behavior of Eq. (26) for different values of μ and t by considering, for simplicity, $\mathcal{V} = 1$, $x' = 0$, $\hbar = 1$, and $\mathcal{D}_\mu = 1$.

for $\gamma = 1$. In particular, we have discussed that the Green function $|\mathcal{G}_\gamma(p, x'; t, t')|$ in the limit $|p| \to \infty$ has a different behavior from the solution obtained for $\gamma = 1$, which was illustrated in Fig. 1. Finally, we expect that the discussion presented here can be useful for the situations connected with the fractional Schrödinger equation.

ACKNOWLEDGMENTS

We thank INCT – SC/CNPq, CAPES, and Fundação Araucária (Brazilian agencies). H.V.R. is especially to Fundação Araucária and CAPES for the financial support Grant No. 113/2013.

APPENDIX: FOX H-FUNCTION

The Fox H function (or H-function) may be defined in terms of the Mellin-Branes type integral[31,37]

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|{}^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] = \mathrm{H}^{m,n}_{p,q}\left[x\left|{}^{(a_1,A_1),\cdots,(a_p,A_p)}_{(b_1,B_1),\cdots,(b_q,B_q)}\right.\right] = \frac{1}{2\pi i}\int_L \chi(\xi)x^{-\xi}d\xi$$

$$\chi(\xi) = \frac{\Pi_{j=1}^{m}\Gamma(b_j - B_j\xi)\Pi_{j=1}^{n}\Gamma(1-a_j+A_j\xi)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j+B_j\xi)\Pi_{j=n+1}^{p}\Gamma(a_j-A_j\xi)}, \quad \text{(A1)}$$

where m, n, p, and q are integers satisfying $0 \le n \le p$ and $1 \le m \le q$. It may also be defined by its Mellin transform

$$\int_0^\infty \mathrm{H}^{m,n}_{p,q}\left[ax\left|{}^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] x^{\xi-1}dx = a^{-\xi}\chi(\xi)\,. \quad \text{(A2)}$$

Here, the parameters have to be defined such that $A_j > 0$ and $B_j > 0$ and $a_j(b_h + \nu) \neq B_h(a_j - \lambda - 1)$ where $\nu, \lambda = 0, 1, 2, \ldots, h = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$. The contour L separates the poles of $\Gamma(b_j - B_j\xi)$ for $j = 1, 2, \ldots, m$ from those of $\Gamma(1 - a_j + A_j\xi)$ for $j = 1, 2, \ldots, n$.[31] The H-function is analytic in x if either (i) $x \neq 0$ and $M > 0$ or (ii) $0 < |x| < 1/B$ and $M = 0$, where $M = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j$ and $B = \prod_{j=1}^{p} A_j^{A_j} \prod_{j=1}^{q} B_j^{-B_j}$.

Some useful properties of the Fox H function found in Ref. 31 are listed below.

(i) The H-function is symmetric in the pairs $(a_1, A_1), \cdots, (a_p, A_p)$, likewise $(a_{n+1}, A_{n+1}), \cdots, (a_p, A_p)$; in $(b_1, B_1), \cdots, (b_q, B_q)$ and in $(b_{n+1}, B_{n+1}), \cdots, (b_q, B_q)$.

(ii) For $k > 0$

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] = k\ \mathrm{H}^{m\ n}_{p\ q}\left[x^k\left|^{(a_p,kA_p)}_{(b_q,kB_q)}\right.\right]. \tag{A3}$$

(iii) The multiplication rule is

$$x^k\ \mathrm{H}^{m,n}_{p,;q}\left[x\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] = \mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_p+kA_p,A_p)}_{(b_q+kB_q,B_q)}\right.\right]. \tag{A4}$$

(iv) For $n \geq 1$ and $q > m$,

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_1,A_1)(a_2,A_2)\cdots(a_p,A_p)}_{(b_1,B_1)\cdots(b_{q-1},B_{q-1})(a_1,A_1)}\right.\right] = \mathrm{H}^{m,n-1}_{p-1,q-1}\left[x\left|^{(a_2,A_2)\cdots(a_p,A_p)}_{(b_1,B_1)\cdots(b_{q-1},B_{q-1})}\right.\right]. \tag{A5}$$

(v) For $m \geq 2$ and $p > n$

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_1,A_1)\cdots(a_{p-1},A_{p-1})(b_1,B_1)}_{(b_1,B_1)(b_2,B_2)\cdots(b_q,B_q)}\right.\right] = \mathrm{H}^{m-1,n}_{p-1,q-1}\left[x\left|^{(a_2,A_2)\cdots(a_{p-1},A_{p-1})}_{(b_2,B_2)\cdots(b_q,B_q)}\right.\right]. \tag{A6}$$

(vi) The relation between the generalized Mittag-Leffler function and the Fox H function is given by

$$\mathrm{E}_{\alpha,\beta}(x) = \mathrm{H}^{1,1}_{1,2}\left[-x\left|^{(0,1)}_{(0,1)(1-\beta,\alpha)}\right.\right]. \tag{A7}$$

(vii) Under Fourier cosine transformation, the H-function transforms as

$$\int_0^\infty \mathrm{H}^{m,n}_{p,q}\left[k\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right]\cos(kx)dx = \frac{\pi}{x}\ \mathrm{H}^{n+1,m}_{q+1,p+2}\left[x\left|^{(1-b_q,B_q),(1,1/2)}_{(1,1),(1-a_p,A_p),(1,1/2)}\right.\right]. \tag{A8}$$

(viii) If the poles of $\prod_{j=1}^m \Gamma(b_j - B_j\xi)$ are simple, the following series expansion is valid

$$\mathrm{H}^{m,n}_{p,q}\left[x\left|^{(a_p,A_p)}_{(b_q,B_q)}\right.\right] = \sum_{h=1}^m\sum_{\nu=0}^\infty \frac{(-1)^\nu x^{(b_h+\nu)/B_h}}{\nu!B_h}\frac{\Pi^m_{j=1,j\neq h}\Gamma\left(b_j - \frac{B_j}{B_h}(b_h+\nu)\right)}{\Pi^q_{j=m+1}\Gamma\left(1-b_j+\frac{B_j}{B_h}(b_h+\nu)\right)}$$
$$\times\frac{\Pi^n_{j=1}\Gamma\left(1-a_j+\frac{A_j}{B_h}(b_h+\nu)\right)}{\Pi^p_{j=n+1}\Gamma\left(a_j-\frac{A_j}{B_h}(b_h+\nu)\right)}. \tag{A9}$$

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[2] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
[3] K. S. Millar and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley & Sons, New York, 1993).
[4] J. T. Machado, V. Kiryakova, and F. Mainardi, Commun. Nonlinear Sci. Numer. Simul. **16**, 1140 (2011).
[5] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, Orlando, 1974).
[6] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2005).
[7] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[8] *Fractional Dynamics: Recent Advances*, edited by J. Klafter, S. C. Lim, and R. Metzler (World Scientific Publishing Company, Singapore, 2011).
[9] R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
[10] Y. Z. Povstenko, Adv. Differ. Equ. **2011**, 930297 (2011).
[11] Y. Z. Povstenko, J. Therm. Stresses **28**, 83 (2004).
[12] Y. Z. Povstenko, J. Math. Sci. **162**, 296 (2009).
[13] T. Kosztołowicz, K. Dworecki, and K. D. Lewandowska, Phys. Rev. E **86**, 021123 (2012).
[14] N. Laskin, Phys. Lett. A **268**, 298 (2000).
[15] N. Laskin, Phys. Rev. E **62**, 3135 (2000).

[16] N. Laskin, Chaos **10**, 780 (2000).
[17] N. Laskin, Phys. Rev. E **66**, 056108 (2002).
[18] M. Naber, J. Math. Phys. **45**, 3339 (2004).
[19] J. Dong, J. Math. Phys. **52**, 042103 (2011).
[20] S. Ş. Bayın, J. Math. Phys. **54**, 012103 (2013).
[21] S. Ş. Bayın, J. Math. Phys. **53**, 042105 (2012).
[22] E. K. Lenzi, H. V. Ribeiro, H. Mukai, and R. S. Mendes, J. Math. Phys. **51**, 092102 (2010).
[23] M. Jeng, S.-L.-Y. Xu, E. Hawkins, and J. M. Schwarz, J. Math. Phys. **51**, 062102 (2010).
[24] H. Ertik, D. Demirhan, H. Sirin, and F. Büyükkiliç, J. Math. Phys. **51**, 082102 (2010).
[25] X. Jiang, H. Qi, and M. Xu, J. Math. Phys. **52**, 042105 (2011).
[26] E. K. Lenzi, B. F. de Oliveira, L. R. da Silva, and L. R. Evangelista, J. Math. Phys. **49**, 032108 (2008).
[27] Y. Luchko, J. Math. Phys. **54**, 012111 (2013).
[28] E. C. de Oliveira and J. Vaz, Jr., J. Phys. A: Math. Theor. **44**, 185303 (2011).
[29] S. I. Muslih, Int. J. Theor. Phys. **49**, 2095 (2010).
[30] E. C. de Oliveira, F. S. Costa, and J. Vaz, Jr., J. Math. Phys. **51**, 123517 (2010).
[31] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function: Theory and Applications* (Springer, New York, 2009).
[32] F. Mainardi, G. Pagnini, and R. K. Saxena, J. Comput. Appl. Math. **178**, 321 (2005).
[33] M. Kleber, Phys. Rep. **236**, 331 (1994).
[34] S. M. Blinder, Phys. Rev. A **37**, 973 (1988).
[35] I. Cacciari and P. Moretti, Phys. Lett. A **359**, 396 (2006).
[36] J. Martins, H. V. Ribeiro, L. R. Evangelista, L. R. da Silva, and E. K. Lenzi, App. Math. Comput. **219**, 2313 (2012).
[37] R. Metzler and T. F. Nonnenmacher, Chem. Phys. **284**, 67 (2002).